T-duality between effective string theories*

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Abstract

We consider the open bosonic string moving in the constant background. We investigate whether the solving of the constraints, obtained by applying the Dirac procedure to the boundary conditions of the open string, which leads to effective closed string theory, and the T-dualization procedure are commutative. We consider the string with mixed boundary conditions. We start by applying the Dirac procedure to these conditions, which results in two parameter dependent constraints. These constraints are solved and for their solution the effective theory is obtained. On the other hand applying the T-dualization procedure to the initial theory one obtains the T-dual theory. As usual, the form of the theory is such that as if the T-dual boundary conditions are already chosen, so that the T-dual coordinates satisfy exactly the opposite set of the boundary conditions then the corresponding coordinates of the initial string. We apply the Dirac procedure to the T-dual boundary conditions, obtain the parameter dependent constraints and solve them to obtain the T-dual effective theory. We show that the effective theories of the initial and T-dual theory remain T-dual, and find the effective T-duality coordinate transformation laws.

1. Introduction

This article is based on our paper [1]. We consider the open string moving in the constant background fields and its T-dual theory. We choose the mixed boundary conditions, for both initial and T-dual coordinates and solve them using techniques developed in [2, 3, 4]. The initial theory with Neumann boundary conditions for coordinate directions x^a and Dirichlet conditions for the rest of the coordinates x^i is by T-dual coordinate transformation laws equivalent to the T-dual theory with dual coordinates y_a satisfying the Dirichlet and y_i the Neumann boundary conditions.

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We treat all boundary conditions as constraints and follow the Dirac procedure. The new constraints are found, first as a Poisson bracket between the hamiltonian and the boundary conditions, and every subsequent as a Poisson bracket between the hamiltonian and the previous constraint. Using the Taylor expansion, we represent this infinite set of constraints we obtain, by only two σ-dependent constraints [5, 6, 7, 8], one for each endpoint. Imposing $2\pi-$periodicity, to the variables building the constraints, one observes that the constraints at $\sigma = \pi$ can be expressed in terms of that at $\sigma = 0$, and that in fact solving one pair of constraints one solves the other pair as well.

The σ-dependent constraints are solved by separating the constraints into their even and odd parts under world-sheet parity transformation $(\Omega : \sigma \to -\sigma)$, and separating the variables building them into their even and odd parts. Solving the σ-dependent constraints, one reduces the phase space by half. Halves of the original canonical variables are treated as effective variables: the independent variables and their canonical conjugates. For the solution of the constraints we obtain the effective theories, defined in terms of the effective variables. We examine their characteristics and confirm that the effective theories of two T-dual theories are also T-dual.

2. The initial and T-dual open bosonic string

Let us consider two T-dual bosonic string theories

$$\kappa \int_\Sigma d\xi^2 \partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu \longleftrightarrow \frac{\kappa^2}{2} \int_\Sigma d\xi^2 \partial_+ y_\mu \Theta_-^{\mu\nu} \partial_- y_\nu, \tag{1}$$

describing the bosonic string and its T-dual moving in constant backgrounds associated with the fields: a metric tensor $G_{\mu\nu}$, a Kalb-Ramond field $B_{\mu\nu}$ and a dilaton field Φ. The integration goes over two-dimensional world-sheet Σ. Σ is usualy parametrized by ξ^α $(\xi^0 = \tau,\ \xi^1 = \sigma)$, but here we use the light-cone coordinates given by

$$\xi^\pm = \frac{1}{2}(\tau \pm \sigma), \qquad \partial_\pm = \partial_\tau \pm \partial_\sigma. \tag{2}$$

$x^\mu(\xi),\ \mu = 0, 1, ..., D-1$ are the coordinates of the D-dimensional space-time, and $y_\mu(\xi)$ are the coordinates of its T-dual. The background field composition is

$$\Pi_{\pm\mu\nu}(x) = B_{\mu\nu}(x) \pm \frac{1}{2} G_{\mu\nu}(x), \tag{3}$$

and the dual background field composition is

$${}^\star\Pi_\pm^{\mu\nu} = \frac{\kappa}{2}\Theta_\mp^{\mu\nu} = -(G_E^{-1}\Pi_\mp G^{-1})^{\mu\nu}, \quad (G_E)_{\mu\nu} = (G - 4BG^{-1}B)_{\mu\nu}, \tag{4}$$

where G_E is the effective metric. The T-dual metric is its inverse

$${}^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}, \tag{5}$$

and a T-dual Kalb-Ramond field is

$$
{}^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}, \tag{6}
$$

where $\theta^{\mu\nu} = -\frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu}$ is the noncommutativity parameter.

The coordinates of the initial and the T-dual theory are related by

$$
\partial_\pm x^\mu \cong -\kappa\Theta_\pm^{\mu\nu}\partial_\pm y_\nu, \quad \partial_\pm y_\mu \cong -2\Pi_{\mp\mu\nu}\partial_\pm x^\nu. \tag{7}
$$

We consider the block diagonal constant metric and Kalb-Ramond field $G_{\mu\nu} = const$, $B_{\mu\nu} = const$

$$
G_{\mu\nu} = \begin{pmatrix} G_{ab} & 0 \\ 0 & G_{ij} \end{pmatrix}, \quad B_{\mu\nu} = \begin{pmatrix} B_{ab} & 0 \\ 0 & B_{ij} \end{pmatrix}, \tag{8}
$$

consequently the composition of the T-dual background fields is also block diagonal $\Theta_\pm^{\mu\nu} = \begin{pmatrix} \Theta_\pm^{ab} & 0 \\ 0 & \Theta_\pm^{ij} \end{pmatrix}$ with components

$$
\begin{aligned}
&\Theta_\pm^{\mu\nu} = \\
&-\frac{2}{\kappa}\begin{pmatrix} ({G_E}^{-1})^{ac} & 0 \\ 0 & ({G_E}^{-1})^{ik} \end{pmatrix}\begin{pmatrix} \Pi_{\pm cd} & 0 \\ 0 & \Pi_{\pm kl} \end{pmatrix}\begin{pmatrix} (G^{-1})^{db} & 0 \\ 0 & (G^{-1})^{lj} \end{pmatrix} \\
&= \begin{pmatrix} \theta^{ab} & 0 \\ 0 & \theta^{ij} \end{pmatrix} \mp \frac{1}{\kappa}\begin{pmatrix} (G_E^{-1})^{ab} & 0 \\ 0 & (G_E^{-1})^{ij} \end{pmatrix}.
\end{aligned} \tag{9}
$$

where $(G_E)_{ab} = G_{ab} - 4B_{ac}(G^{-1})^{cd}B_{db}$, $(G_E)_{ij} = G_{ij} - 4B_{ik}(G^{-1})^{kl}B_{lj}$ and $\theta^{ab} = -\frac{2}{\kappa}(G_E^{-1})^{ac}\,B_{cd}(G^{-1})^{db}$, $\theta^{ij} = -\frac{2}{\kappa}(G_E^{-1})^{ik}\,B_{kl}(G^{-1})^{lj}$.

2.1. Boundary conditions

We will consider the boundary conditions of the initial theory

$$
\gamma_\mu^{(0)}\delta x^\mu\Big|_0^\pi = 0, \quad \gamma_\mu^{(0)} = \kappa\big(\Pi_{+\mu\nu}\partial_- x^\nu + \Pi_{-\mu\nu}\partial_+ x^\nu\big). \tag{10}
$$

For each of the space-time coordinates of the open bosonic string one can fulfill the boundary conditions (10) by choosing either the Neumann or the Dirichlet boundary condition. Let us choose the Neumann condition for coordinates x^a, $a = 0, 1, \ldots, p$ and the Dirichlet condition for coordinates x^i, $i = p+1, \ldots, D-1$, which read

$$
\begin{aligned}
&\textit{Neumann}: \quad \gamma_a^{(0)}\Big|_{\partial\Sigma} = 0, \quad {}_N\gamma_a^0 \equiv \gamma_a^{(0)} = \kappa\big(\Pi_{+ab}\partial_- x^b + \Pi_{-ab}\partial_+ x^b\big), \\
&\textit{Dirichlet}: \quad \kappa\dot{x}^i\Big|_{\partial\Sigma} = 0, \quad {}_D\gamma_0^i \equiv \kappa\dot{x}^i.
\end{aligned} \tag{11}
$$

The boundary conditions of the T-dual string are

$$ {}^\star\gamma^{(0)\mu}\,\delta y_\mu\Big|_0^\pi = 0, \quad {}^\star\gamma^{(0)\mu} = \frac{\kappa^2}{2}\Big[\Theta_-^{\mu\nu}\partial_- y_\nu + \Theta_+^{\mu\nu}\partial_+ y_\nu\Big]. \tag{12} $$

The T-dual theory is equivalent to a open string theory (1) with chosen boundary conditions (11), if the T-dual boundary conditions are fulfilled in a Neumann way for coordinates y_i and in a Dirichlet way for y_a

$$ \begin{aligned} &\textit{Neumann}: \ {}^\star\gamma^{(0)i}\Big|_{\partial\Sigma} = 0, \quad {}^\star_N\gamma_0^i \equiv {}^\star\gamma^{(0)i} = \frac{\kappa^2}{2}\Big[\Theta_-^{ij}\partial_- y_j + \Theta_+^{ij}\partial_+ y_j\Big], \\ &\textit{Dirichlet}: \ \kappa\dot{y}_a\Big|_{\partial\Sigma} = 0, \quad {}^\star_D\gamma_a^0 \equiv \kappa\dot{y}_a. \end{aligned} \tag{13} $$

This is because of the T-duality transformation law (7), which gives

$$ -\kappa\dot{x}^\mu \cong {}^\star\gamma^{(0)\mu}(y), \quad \gamma_\mu^{(0)}(x) \cong -\kappa\dot{y}_\mu, \tag{14} $$

and consequently

$$ \begin{aligned} {}_D\gamma_0^i &\equiv \kappa\dot{x}^i \cong -{}^\star\gamma^{(0)i} \equiv -{}^\star_N\gamma_0^i, \\ {}_N\gamma_a^0 &\equiv \gamma_a^{(0)} \cong -\kappa\dot{y}_a = -{}^\star_D\gamma_a^0. \end{aligned} \tag{15} $$

So, T-dualization interchanges the Dirichlet and Neumann boundary conditions.

3. Canonical form of the boundary conditions and their consistency by Dirac procedure

We will treat the T-dual boundary conditions (11) and (13), of theories (1) as constraints and we will apply the Dirac consistency procedure, following [2, 3]. This will be done in a canonical description, because the conditions can be expressed in terms of the currents building the energy-momentum tensors, and consequently the hamiltonians. So, to calculate new constraints one only uses the algebra of these currents.

The canonical hamiltonian densities of theories (1) are

$$ \mathcal{H}_C = T_- - T_+ = \frac{1}{4\kappa}(G^{-1})^{\mu\nu}\Big[j_{+\mu}j_{+\nu} + j_{-\mu}j_{-\nu}\Big], \tag{16} $$

and

$$ {}^\star\mathcal{H}_C = {}^\star T_- - {}^\star T_+ = \frac{1}{4\kappa}({}^\star G^{-1})^{\mu\nu}\Big[{}^\star j_{+\mu}{}^\star j_{+\nu} + {}^\star j_{-\mu}{}^\star j_{-\nu}\Big], \tag{17} $$

where $T_\pm = \mp\frac{1}{4\kappa}(G^{-1})^{\mu\nu}j_{\pm\mu}j_{\pm\nu}$ and ${}^\star T_\pm = \mp\frac{1}{4\kappa}({}^\star G^{-1})^{\mu\nu}\,{}^\star j_{\pm\mu}{}^\star j_{\pm\nu}$ are the energy-momentum tensors. The currents are given by

$$ j_{\pm\mu} = \pi_\mu + 2\kappa\Pi_{\pm\mu\nu}x'^\nu, \quad {}^\star j_\pm^\mu = {}^\star\pi^\mu + 2\kappa{}^\star\Pi_\pm^{\mu\nu}y'_\nu, \tag{18} $$

where the momenta are $\pi_\mu = -2\kappa B_{\mu\nu} x'^\nu + \kappa G_{\mu\nu}\dot{x}^\nu$ and ${}^\star\pi^\mu = -\kappa^2\theta^{\mu\nu} y'_\nu + \kappa(G_E^{-1})^{\mu\nu}\dot{y}_\nu = -2\kappa\,{}^\star B^{\mu\nu} y'_\nu + \kappa\,{}^\star G^{\mu\nu}\dot{y}_\nu$. Using the last relations the currents can be rewritten in terms of coordinates as

$$j_{\pm\mu} = \kappa G_{\mu\nu}\partial_\pm x^\nu, \quad {}^\star j_\pm^\mu = \kappa(G_E^{-1})^{\mu\nu}\partial_\pm y_\nu. \tag{19}$$

The algebra of the initial currents [9] in a constant background is given by

$$\begin{aligned}\big\{j_{\pm\mu}(\sigma), j_{\pm\nu}(\bar\sigma)\big\} &= \pm\, 2\kappa\, G_{\mu\nu}\,\delta'(\sigma-\bar\sigma),\\ \big\{j_{\pm\mu}(\sigma), j_{\mp\nu}(\bar\sigma)\big\} &= 0.\end{aligned} \tag{20}$$

The initial Neumann (N) and Dirichlet (D) boundary conditions (11) are in a canonical form given by

$$\begin{aligned}{}_N\gamma_a^0 &= \Pi_{+ab}(G^{-1})^{bc} j_{-c} + \Pi_{-ab}(G^{-1})^{bc} j_{+c},\\ {}_D\gamma_0^i &= \kappa\dot{x}^i = \frac{1}{2}(G^{-1})^{ij}(j_{+j}+j_{-j}).\end{aligned} \tag{21}$$

The T-dual conditions in a canonical form are

$$\begin{aligned}{}^\star_N\gamma_0^i &= {}^\star\Pi_{+ij}({}^\star G^{-1})^{jk}\,{}^\star j_{-k} +{}^\star \Pi_{-ij}({}^\star G^{-1})^{jk}\,{}^\star j_{+k},\\ {}^\star_D\gamma_a^0 &= \frac{1}{2}({}^\star G^{-1})_{ab}\Big({}^\star j_+^b + {}^\star j_-^b\Big).\end{aligned} \tag{22}$$

Although the Neumann boundary conditions transform into Dirichlet conditions of the T-dual theory by (15), and Dirichlet's to Neumann, one can observe that the form of Neumann and Dirichlet conditions has not changed in T-dual theories.

Following the Dirac procedure, one can impose consistency to these constraints. The additional constraints are defined for every $n \geq 1$ by

$${}_N\gamma_a^n = \{H_C, {}_N\gamma_a^{n-1}\}, \quad {}_D\gamma_n^i = \{H_C, {}_D\gamma_{n-1}^i\}, \tag{23}$$

with $H_C = \int d\sigma\mathcal{H}_C$ being the canonical hamiltonian of the initial theory, and similarly for the T-dual theory. All the constraints can be gathered into only two σ dependent constraints. They are obtained by multiplying every constraint by an appropriate degree of the world-sheet space parameter σ and adding the terms together

$$\Gamma_a^N(\sigma) = \sum_{n\geq 0}\frac{\sigma^n}{n!}\,{}_N\gamma_a^n\Big|_{\sigma=0}, \quad \Gamma_D^i(\sigma) = \sum_{n\geq 0}\frac{\sigma^n}{n!}\,{}_D\gamma_n^i\Big|_{\sigma=0}. \tag{24}$$

We obtain the explicit form of the sigma dependent constraints of the initial theory

$$\begin{aligned}\Gamma_a^N(\sigma) &= \Pi_{+ab}(G^{-1})^{bc} j_{-c}(\sigma) + \Pi_{-ab}(G^{-1})^{bc} j_{+c}(-\sigma),\\ \Gamma_D^i(\sigma) &= \frac{1}{2}(G^{-1})^{ij}\Big[j_{+j}(-\sigma) + j_{-j}(\sigma)\Big],\end{aligned} \tag{25}$$

and of its T-dual

$$\begin{aligned} {}^\star\Gamma^i_N(\sigma) &= \frac{\kappa}{2}\Big[\Theta^{ij}_- G^E_{jk}\,{}^\star j^k_-(\sigma) + \Theta^{ij}_+ G^E_{jk}\,{}^\star j^k_+(-\sigma)\Big], \\ {}^\star\Gamma^D_a(\sigma) &= \frac{1}{2}G^E_{ab}\big[{}^\star j^b_+(-\sigma) + {}^\star j^b_-(\sigma)\big]. \end{aligned} \tag{26}$$

One can check that the σ-dependent constraints are of the second class and one therefore can solve them.

Obviously, the parameter dependent constraints are given in terms of currents depending on either σ or $-\sigma$. So, it is useful to express the canonical variables into their even and odd parts, with respect to σ. For the initial coordinates one has

$$x^\mu = q^\mu + \bar q^\mu, \quad q^\mu = \sum_{n\geq 0}\frac{\sigma^{2n}}{(2n)!}\,x^{(2n)\mu}\Big|_{\sigma=0}, \quad \bar q^\mu = \sum_{n\geq 0}\frac{\sigma^{2n+1}}{(2n+1)!}\,x^{(2n+1)\mu}\Big|_{\sigma=0}, \tag{27}$$

and for the momenta one has

$$\pi_\mu = p_\mu + \bar p_\mu, \quad p_\mu = \sum_{n\geq 0}\frac{\sigma^{2n}}{(2n)!}\,\pi^{(2n)}_\mu\Big|_{\sigma=0}, \quad \bar p_\mu = \sum_{n\geq 0}\frac{\sigma^{2n+1}}{(2n+1)!}\,\pi^{(2n+1)}_\mu\Big|_{\sigma=0}. \tag{28}$$

Similarly, we separate the T-dual variables into the odd and even parts

$$\begin{aligned} y_\mu &= k_\mu + \bar k_\mu, \\ {}^\star\pi^\mu &= {}^\star p^\mu + {}^\star\bar p^\mu. \end{aligned} \tag{29}$$

After this separations the constraints become

$$\begin{aligned} \Gamma^N_a(\sigma) &= 2(BG^{-1})_a{}^b\, p_b + \bar p_a - \kappa(G_E)_{ab}\bar q'^b, \\ \Gamma^i_D(\sigma) &= (G^{-1})^{ij}\Big[p_j - \kappa G_{jk}q'^k + 2\kappa B_{jk}\bar q'^k\Big], \end{aligned} \tag{30}$$

and

$$\begin{aligned} {}^\star\Gamma^i_N(\sigma) &= -2(G^{-1}B)^i{}_j\,{}^\star p^j + {}^\star\bar p^i - \kappa(G^{-1})^{ij}\bar k'_j, \\ {}^\star\Gamma^D_a(\sigma) &= G^E_{ab}\Big[{}^\star p^b + \kappa^2\theta^{bc}\bar k'_c - \kappa(G^{-1}_E)^{bc}k'_c\Big]. \end{aligned} \tag{31}$$

This form of constraints is solvable and has a solution

$$\begin{aligned} \bar p_a &= 0, \qquad \bar q'^a = -\theta^{ab}p_b, \\ q'^i &= 0, \qquad p_i = -2\kappa B_{ij}\bar q'^j, \end{aligned} \tag{32}$$

and in T-dual theory

$$\begin{aligned} {}^\star\bar p^i &= 0, \qquad \bar k'_i = -\frac{2}{\kappa}B_{ij}\,{}^\star p^j, \\ {}^\star p^a &= -\kappa^2\theta^{ab}\bar k'_b, \qquad k'_a = 0. \end{aligned} \tag{33}$$

Solving the constraints has reduced the phase space by half. The σ-derivative of coordinates and the momenta for the solution (32) are

$$x'^{\mu} = \begin{cases} q'^{a} - \theta^{ab} p_b, & \mu=a, \\ \\ \bar{q}'^{i}, & \mu=i, \end{cases} \tag{34}$$

and

$$\pi_{\mu} = \begin{cases} p_a, & \mu=a, \\ \\ \bar{p}_i - 2\kappa B_{ij} \bar{q}'^{j}, & \mu=i. \end{cases} \tag{35}$$

The T-dual coordinate σ-derivative and the T-dual momenta for the solution (33) are

$$y'_{\mu} = \begin{cases} \bar{k}'_a, & \mu=a, \\ \\ k'_i - \frac{2}{\kappa} B_{ij} {}^{\star}p^{j}, & \mu=i, \end{cases} \tag{36}$$

and

$${}^{\star}\pi^{\mu} = \begin{cases} {}^{\star}\bar{p}^{a} - \kappa^2 \theta^{ab} \bar{k}'_b, & \mu=a, \\ \\ {}^{\star}p^{i}, & \mu=i. \end{cases} \tag{37}$$

When one considers the constraints at the other string end-point $\sigma = \pi$, one multiplies every constraint with the appropriate power of $\sigma - \pi$ and sums the products into ${}_{\pi}\Gamma^{N}_{a}(\sigma)$ and ${}_{\pi}\Gamma^{i}_{D}(\sigma)$. The sigma dependent constraints at π and 0 differ by the following exchange

$$j_{+a}(2\pi - \sigma) \longleftrightarrow j_{+a}(-\sigma), \quad j_{+i}(2\pi - \sigma) \longleftrightarrow j_{+i}(-\sigma). \tag{38}$$

It follows that the extension of a domain [2] of the variables building the currents, i.e. original coordinates and momenta and demand their 2π-periodicity $x^{\mu}(\sigma+2\pi) = x^{\mu}(\sigma)$, and $\pi_{\mu}(\sigma+2\pi) = \pi_{\mu}(\sigma)$ makes the solutions (32) and (33) the universal solution for both $\sigma-$dependent constraints at $\sigma = 0, \pi$.

4. Effective theories for the solution of boundary conditions

Next, we will substitute the solution of the constraints into the canonical hamiltonians, to obtain the effective hamiltonians. Using the equations of motion for momenta, we will find the corresponding effective lagrangians. Effective theories describe the closed effective strings.

$$\begin{array}{ccc} \kappa \int d\xi^2 \partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu & \overset{T}{\longleftrightarrow} & \frac{\kappa^2}{2} \int d\xi^2 \partial_+ y_\mu \Theta_-^{\mu\nu} \partial_- y_\nu \\ \Big\downarrow{\scriptstyle \Gamma=0} & & \Big\downarrow{\scriptstyle {}^\star\Gamma=0} \\ ? & & ? \end{array}$$

The solution of the initial boundary conditions is such that the effective hamiltonian will be given in terms of odd $\bar{q}^i, \bar{p}_i$ in Dirichlet directions and even q^a, p_a in Neumann directions. It reads

$$\mathcal{H}^{eff} = \mathcal{H}_N^{eff}(q^a, p_a) + \mathcal{H}_D^{eff}(\bar{q}^i, \bar{p}_i), \tag{39}$$

where

$$\mathcal{H}_N^{eff}(q^a, p_a) = \frac{\kappa}{2}\, q'^a G^E_{ab} q'^b + \frac{1}{2\kappa} p_a (G_E^{-1})^{ab} p_b,$$
$$\mathcal{H}_D^{eff}(\bar{q}^i, \bar{p}_i) = \frac{\kappa}{2}\, \bar{q}'^i G_{ij} \bar{q}'^j + \frac{1}{2\kappa} \bar{p}_i (G^{-1})^{ij} \bar{p}_j. \tag{40}$$

Simmilarlly, the effective T-dual hamiltonian is given in terms of odd $\bar{k}_a, {}^\star\bar{p}^a$ in Dirichlet directions and even $k_i, {}^\star p^i$ in Neumann directions. The corresponding effective lagrangians will consequently depend on both even and odd coordinate parts $q^a, \bar{q}^i$ and $k_i, \bar{k}_a$. The effective T-dual hamiltonian is

$${}^\star\mathcal{H}^{eff} = {}^\star\mathcal{H}_D^{eff}(\bar{k}_a, {}^\star\bar{p}^a) + {}^\star\mathcal{H}_N^{eff}(k_i, {}^\star p^i), \tag{41}$$

where

$${}^\star\mathcal{H}_D^{eff}(\bar{k}_a, {}^\star\bar{p}^a) = \frac{\kappa}{2}\, \bar{k}'_a (G_E^{-1})^{ab} \bar{k}'_b + \frac{1}{2\kappa} {}^\star\bar{p}^a (G_E)_{ab} {}^\star\bar{p}^b,$$
$${}^\star\mathcal{H}_N^{eff}(k_i, {}^\star p^i) = \frac{\kappa}{2}\, k'_i (G^{-1})^{ij} k'_j + \frac{1}{2\kappa} {}^\star p^i G_{ij} {}^\star p^j. \tag{42}$$

The effective Lagrangians of the above effective hamiltonians are

$$\mathcal{L}^{eff} = \Big[\pi_\mu \dot{x}^\mu\Big]\Big|_{\Gamma_\mu=0} - \mathcal{H}^{eff}, \quad {}^\star\mathcal{L}^{eff} = \Big[{}^\star\pi^\mu \dot{y}_\mu\Big]\Big|_{{}^\star\Gamma_\mu=0} - {}^\star\mathcal{H}^{eff}. \tag{43}$$

The effective lagrangians can be separated into

$$\mathcal{L}^{eff} = \mathcal{L}_N(q,p) + \mathcal{L}_D(\bar{q},\bar{p}), \quad {}^\star\mathcal{L}^{eff} = {}^\star\mathcal{L}_D(\bar{k}, {}^\star\bar{p}) + {}^\star\mathcal{L}_N(k, {}^\star p), \tag{44}$$

with

$$\mathcal{L}_N(q,p) = p_a \dot{q}^a - \mathcal{H}_N^{eff}(q^a, p_a), \quad \mathcal{L}_D(\bar{q},\bar{p}) = \bar{p}_i \dot{\bar{q}}^i - \mathcal{H}_D^{eff}(\bar{q}^i, \bar{p}_i),$$
$${}^\star\mathcal{L}_D(\bar{k}, {}^\star\bar{p}) = {}^\star\bar{p}^a \dot{\bar{k}}_a - {}^\star\mathcal{H}_D^{eff}(\bar{k}_a, {}^\star\bar{p}^a), \quad {}^\star\mathcal{L}_N(k, {}^\star p) = {}^\star p_i \dot{k}_i - {}^\star\mathcal{H}_N^{eff}(k_i, {}^\star p^i). \tag{45}$$

The explicit forms of the effective lagrangians are found by eliminating the momenta from (45), using the equations of motion for them

$$p_a = \kappa G^E_{ab}\dot{q}^b, \quad \bar{p}_i = \kappa G_{ij}\dot{\bar{q}}^j, \tag{46}$$

and

$$^\star\bar{p}^a = \kappa (G_E^{-1})^{ab}\dot{\bar{k}}_b, \quad ^\star p^i = \kappa (G^{-1})^{ij}\dot{k}_j. \tag{47}$$

For these equations the σ-derivatives of the initial and T-dual coordinates, given by (34) and (36), become

$$x'^\mu = \begin{cases} q'^a + 2(G^{-1}B)^a{}_b\,\dot{q}^b, & \mu=a, \\ \\ \bar{q}'^i, & \mu=i, \end{cases} \tag{48}$$

and

$$y'_\mu = \begin{cases} \bar{k}'_a, & \mu=a, \\ \\ k'_i - 2(BG^{-1})_i{}^j\,\dot{k}_j, & \mu=i. \end{cases} \tag{49}$$

In order to find the expression for the initial and the T-dual coordinate we need to introduce a double coordinate $\tilde{q}^a$ of the even part of the initial coordinate q^a

$$\dot{\tilde{q}}^a = q'^a, \quad \tilde{q}'^a = \dot{q}^a, \tag{50}$$

and a double coordinate $\tilde{k}_i$ of the even part of the T-dual coordinate k_i

$$\dot{\tilde{k}}_i = k'_i, \quad \tilde{k}'_i = \dot{k}_i. \tag{51}$$

The coordinates become

$$x^\mu = \begin{cases} q^a + 2(G^{-1}B)^a{}_b\,\tilde{q}^b, & \mu=a, \\ \\ \bar{q}^i, & \mu=i, \end{cases} \tag{52}$$

and

$$y_\mu = \begin{cases} \bar{k}_a, & \mu=a, \\ \\ k_i - 2(BG^{-1})_i{}^j\,\tilde{k}_j, & \mu=i. \end{cases} \tag{53}$$

So, after elimination of the momenta the effective lagrangians become

$$\mathcal{L}^{eff} = \mathcal{L}_N(q) + \mathcal{L}_D(\bar{q}), \quad ^\star\mathcal{L}^{eff} = {}^\star\mathcal{L}_D(\bar{k}) + {}^\star\mathcal{L}_N(k), \tag{54}$$

where the lagrangians (45) reduced to

$$\mathcal{L}_N(q) = \frac{\kappa}{2} G^E_{ab}\, \eta^{\alpha\beta}\partial_\alpha q^a \partial_\beta q^b, \qquad \mathcal{L}_D(\bar{q}) = \frac{\kappa}{2} G_{ij}\, \eta^{\alpha\beta}\partial_\alpha \bar{q}^i \partial_\beta \bar{q}^j,$$
$$^\star\mathcal{L}_D(\bar{k}) = \frac{\kappa}{2}(G_E^{-1})^{ab}\, \eta^{\alpha\beta}\partial_\alpha \bar{k}_a \partial_\beta \bar{k}_b, \qquad ^\star\mathcal{L}_N(k) = \frac{\kappa}{2}(G^{-1})^{ij}\, \eta^{\alpha\beta}\partial_\alpha k_i \partial_\beta k_j. \tag{55}$$

5. T-duality between effective theories

Introducing the effective coordinates and T-dual effective coordinates

$$Q^\mu = \begin{bmatrix} q^a \\ \bar{q}^i \end{bmatrix}, \quad K_\mu = \begin{bmatrix} \bar{k}_a \\ k_i \end{bmatrix}, \tag{56}$$

and their corresponding canonically conjugated momenta

$$P_\mu = \begin{bmatrix} p_a \\ \bar{p}_i \end{bmatrix}, \quad ^\star P^\mu = \begin{bmatrix} ^\star\bar{p}^a \\ ^\star p^i \end{bmatrix}, \tag{57}$$

one can rewrite the effective hamiltonians (39) and (41) as

$$\mathcal{H}^{eff} = \frac{\kappa}{2} Q'^\mu G^{eff}_{\mu\nu} Q'^\nu + \frac{1}{2\kappa} P_\mu (G^{-1}_{eff})^{\mu\nu} P_\nu,$$
$$^\star\mathcal{H}^{eff} = \frac{\kappa}{2} K'_\mu {}^\star G^{\mu\nu}_{eff} K'_\nu + \frac{1}{2\kappa} {}^\star P^\mu ({}^\star G^{-1}_{eff})_{\mu\nu} {}^\star P^\nu, \tag{58}$$

where

$$G^{eff}_{\mu\nu} = \begin{pmatrix} G^E_{ab} & 0 \\ 0 & G_{ij} \end{pmatrix}, \qquad ^\star G^{\mu\nu}_{eff} = \begin{pmatrix} (G_E^{-1})^{ab} & 0 \\ 0 & (G^{-1})^{ij} \end{pmatrix}. \tag{59}$$

The T-duality relations

$$\kappa x'^\mu \cong {}^\star\pi^\mu, \quad \kappa y'_\mu \cong \pi_\mu, \tag{60}$$

for the solution of the boundary conditions and separating the odd and even parts reduce to

$$\kappa q'^a \cong {}^\star\bar{p}^a, \quad \kappa \bar{k}'_a \cong p_a,$$
$$\kappa \bar{q}'^i \cong {}^\star p^i, \quad \kappa k'_i \cong \bar{p}_i, \tag{61}$$

which gives

$$\kappa Q'^\mu \cong {}^\star P^\mu, \quad \kappa K'_\mu \cong P_\mu. \tag{62}$$

Additionally, the background fields (59) are T-dual as expected, because by T-duality the metric should transform to the inverse of the effective metric.

In absence of the effective Kalb-Ramond field this means the T-dual metric should be inverse to the initial metric, what is just the case

$$(G^{eff}_{\mu\nu})^{-1} = \begin{pmatrix} G^E_{ab} & 0 \\ 0 & G_{ij} \end{pmatrix}^{-1} = \begin{pmatrix} (G^{-1}_E)^{ab} & 0 \\ 0 & (G^{-1})^{ij} \end{pmatrix} = {}^\star G^{\mu\nu}_{eff}. \tag{63}$$

Using (62) and (63) we can conclude that the effective hamiltonians (58) are T-dual to each other.

The corresponding lagrangians (54) are given by

$$\mathcal{L}^{eff} = \dot{Q}^\mu P_\mu - \mathcal{H}^{eff}(Q,P), \quad {}^\star\mathcal{L}^{eff} = \dot{K}_\mu {}^\star P^\mu - {}^\star\mathcal{H}^{eff}(K, {}^\star P), \tag{64}$$

which for the equations of motion for momenta (46) and (47)

$$P_\mu = \kappa G^{eff}_{\mu\nu}\dot{Q}^\nu, \quad {}^\star P_\mu = \kappa\, {}^\star G^{eff}_{\mu\nu}\dot{K}_\nu, \tag{65}$$

become

$$\mathcal{L}^{eff} = \frac{\kappa}{2}\eta^{\alpha\beta}\,\partial_\alpha Q^\mu\, G^{eff}_{\mu\nu}\,\partial_\beta Q^\nu, \quad {}^\star\mathcal{L}^{eff} = \frac{\kappa}{2}\eta^{\alpha\beta}\,\partial_\alpha K_\mu\, {}^\star G^{\mu\nu}_{eff}\,\partial_\beta K_\nu. \tag{66}$$

Combining (62) with (65) one obtains

$$Q'^\mu \cong {}^\star G^{\mu\nu}_{eff}\dot{K}_\nu, \quad K'_\mu \cong G^{eff}_{\mu\nu}\dot{Q}^\nu. \tag{67}$$

Therefore, the effective and T-dual effective variables Q^μ and K_μ are connected by

$$\partial_\pm K_\mu \cong \pm G^{eff}_{\mu\nu}\partial_\pm Q^\nu. \tag{68}$$

This is the T-dual effective coordinate transformation law. Using it together with (63), one can conclude that the effective lagrangians (66) are T-dual. This law is in agreement with the T-dual coordinate transformation law (7), for $B_{\mu\nu} = 0$

$$\partial_\pm y_\mu \cong \pm G_{\mu\nu}\partial_\pm x^\nu, \tag{69}$$

keeping in mind that the metric is replaced by the effective metric $G_{\mu\nu} \to G^{eff}_{\mu\nu}$.

6. Conclusion

In the present paper we showed that the T-dualization procedure and the solving of the mixed boundary conditions, treated as constraints in the Dirac consistency procedure, do commute. The results are summarized in the following diagram

$$
\begin{array}{ccc}
\kappa\int d\xi^2\partial_+x^\mu\Pi_{+\mu\nu}\partial_-x^\nu & \overset{T}{\longleftrightarrow} & \frac{\kappa^2}{2}\int d\xi^2\partial_+y_\mu\Theta_-^{\mu\nu}\partial_-y_\nu \\
\Big\downarrow{\scriptstyle \Gamma=0} & & \Big\downarrow{\scriptstyle {}^\star\Gamma=0} \\
\frac{\kappa}{2}\int d\xi^2\partial_+Q^\mu G^{eff}_{\mu\nu}\partial_-Q^\nu & \overset{T}{\longleftrightarrow} & \frac{\kappa}{2}\int d\xi^2\partial_+K_\mu{}^\star G^{\mu\nu}_{eff}\partial_-K_\nu.
\end{array}
$$

We started by considering the string described by the open string sigma model (left up). The string is moving in the constant metric $G_{\mu\nu}$ and a constant Kalb-Ramond field $B_{\mu\nu}$. We chose the Neumann boundary conditions for some directions x^a and the Dirichlet boundary conditions for all other directions x^i. We treated these conditions as canonical constraints, and applied the Dirac procedure. The boundary conditions where rewritten in a canonical form in terms of currents building the energy-momentum tensor components. By Dirac procedure the new constraints are found commuting the hamiltonian with the known constraints. The canonical form of constraints allowed us a simple calculation of the exact form of the infinitely many constraints. From these constraints we formed two σ-dependent constraints, for every string endpoint, by multiplying every obtained constraint with the appropriate power of σ for constraints in $\sigma=0$ and power of $\pi-\sigma$ for constraints in $\sigma=\pi$, and adding these terms together into Taylor expansions. The constraints at $\sigma=0$ and $\sigma=\pi$ were found to be equivalent by imposing 2π-periodicity condition for the canonical variables x^μ and π_μ.

The σ-dependent constraints are of the second class. To solve them we introduced even and odd parts of the initial canonical variables. We found the solution and expressed the σ-derivative of the initial coordinate x'^μ and the initial momentum π_μ in terms of even parts q^a, p_a of x^μ, π_μ in Neumann directions and of their odd parts $\bar{q}^i, \bar{p}_i$ in Dirichlet directions, see (34) and (35.) For the solution of constraints $\Gamma=0$, the theory reduced to the effective theory (left down). We obtained the effective hamiltonian (39). For the equations of motion for momenta, we obtained the corresponding effective lagrangian (54).

We also considered the T-dual (right up) of the initial theory. We applied the Dirac procedure to the mixed boundary conditions of the T-dual theory. The constraints where solved, which reduced the phase space to $\bar{k}_a, {}^\star\bar{p}^a$ in D-sector and $k_i, {}^\star p^i$ in N-sector. For the solution of T-dual constraints ${}^\star\Gamma=0$, we obtained the T-dual effective hamiltonian (41), as well as the corresponding T-dual effective lagrangian (54) (right down).

The canonically conjugated effective variables are now pairs q^a, p_a and $\bar{q}^i, \bar{p}_i$ for the initial and $k_i, {}^\star p^i$ and $\bar{k}_a, {}^\star\bar{p}^a$ for the T-dual effective theory. The effective variables in both effective theories satisfy the modified Poisson brackets. This is different, in comparison to the choice of the Neumann boundary conditions for all directions [2, 3, 4]. In that case, solving the constraints leads to full elimination of odd variables. For constant initial background fields, considered in this paper the effective fields are constant. But, the nongeometricity obtained earlier in [2, 3, 4] can still be seen. It appears in a fact that coordinates of the initial and T-dual theories, can not be expressed without an introduction of double coordinates, see (52) and (53).

The obtained effective theories, defined in terms of the effective variables, where compared using the T-dualization procedure. It was confirmed that the corresponding background fields (the effective metrics $G^{eff}_{\mu\nu}$ and ${}^\star G^{\mu\nu}_{eff}$ (59)) are T-dual to each other. Also, the effective variables of the initial effective theory are confirmed to be T-dual to the T-dual effective variables of the T-dual effective theory, by obtaining the T-duality law connecting them. This law was an appropriate reduction of the standard T-duality coordinate transformation law. Therefore, we showed the T-duality of the reduced bosonic string theories.

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