XYZ Gaudin model with boundary terms*

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Abstract

We study the elliptic Gaudin model as a quasi-classical limit of the XYZ Heisenberg spin chain with the most general K-matrix. In particular, we give the generating function of the Gaudin Hamiltonians with boundary terms.

1. Introduction

There is extensive literature on the subject of Gaudin algebras, discussing its various generalizations in many diferent context [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43]. Here we derive the generating function of the Gaudin Hamiltonians with boundary terms in the elliptic case following Sklyanin's approach in the periodic case [4], as we have done for the rational and trigonometric Gaudin model [44, 45, 43, 46, 47]. In the section 2. we review the XYZ Heisenberg spin chain. Our starting point is

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the corresponding R-matrix [48, 49] and its most relevant properties. The relevant Lax operator as well as the definition of the Sklyanin algebra [50] is our next step. Then we study the properties of the central element of the RLL-relations, the so-called quantum determinant of the Lax operator, and the representation theory of the Sklyanin algebra which follows from the fusion procedure [50, 51, 52]. Sklyanin showed how to introduce non-periodic boundary conditions which are compatible with the integrability of the bulk system[27]. Following his approach we study the general solution of the relevant Reflection Equation [53, 54, 55] which at the same time defines the corresponding solution of the so-called dual Reflection Equation. The Reflection Equation Algebra provides the suitable algebraic framework for studying the non-periodic XYZ Heisenberg spin chain. This algebra is generated by the entries of the so-called Sklyanin monodromy matrix. As Sklyanin showed [27], the trace of the product of this monodromy matrix with the solution of the dual Reflection Equation yields the transfer matrix of the chain, i.e. the generator of an Abelian subalgebra which is relevant for the dynamics of the system. Moreover, the Sklyanin monodromy matrix admits a central element of the Reflection Equation Algebra. In the section 3. we study the non-periodic elliptic Gaudin model as the quasi-classical limit of the XYZ Heisenberg spin chain. Following Sklyanin's approach in the rational case [4] we show how the linear combination of the quantum determinant of the elliptic monodromy matrix together with its trace yields the generation function of the elliptic Gaudin model, in the periodic case. To study the non-periodic elliptic Gaudin model we follow the same approach we have used successfully in the rational [44] as well as in the trigonometric case [47]. Namely, we combine the classical r-matrix together with the general solution of the classical Reflection Equation into the so-called non-unitary classical r-matrix which is the key notion in the study of the non-periodic Gaudin model. The relevant Lax operator can then be defined in somewhat natural way [44, 45, 46, 47]. This Lax operator and the non-unitary classical r-matrix determine the linear bracket which is antisymmetric and satisfies the Jacobi identity, defining the dynamical symmetry algebra which is of utmost importance for the solvability of the system. The generating function of the Gaudin Hamiltonian is then introduced in a similar way like for the other non-periodic Gaudin models [44, 45, 46, 47]. Finally, in the appendix A we summarise some fundamental identities satisfied by the Jacobi elliptic functions used in the text.

2. XYZ Heisenberg spin chain

The R-matrix relevant for the XYZ Heisenberg spin chain is given by [48, 49]

$$R(\lambda, \eta, \kappa) = \mathbb{1} + \sum_{\alpha=1}^{3} W_\alpha(\lambda, \eta, \kappa)\ \sigma^\alpha \otimes \sigma^\alpha, \tag{1}$$

where

$$
\begin{aligned}
W_1(\lambda,\eta,\kappa) =& \frac{\mathrm{cn}(\lambda+\eta,\kappa)\ \mathrm{sn}(\eta,\kappa)}{\mathrm{sn}(\lambda+\eta,\kappa)\ \mathrm{cn}(\eta,\kappa)},\\
W_2(\lambda,\eta,\kappa) =& \frac{\mathrm{dn}(\lambda+\eta,\kappa)\ \mathrm{sn}(\eta,\kappa)}{\mathrm{sn}(\lambda+\eta,\kappa)\ \mathrm{dn}(\eta,\kappa)},\\
W_3(\lambda,\eta,\kappa) =& \frac{\mathrm{sn}(\eta,\kappa)}{\mathrm{sn}(\lambda+\eta,\kappa)},
\end{aligned} \tag{2}
$$

the functions $\mathrm{sn}(\lambda,\kappa)$, $\mathrm{cn}(\lambda,\kappa)$, and $\mathrm{dn}(\lambda,\kappa)$ are the usual Jacobi elliptic functions, λ is a spectral parameter, η is a quasi-classical parameter, κ is the modulus and σ^α, $\alpha = 1,2,3$, are the Pauli matrices

$$
\sigma^\alpha = \begin{pmatrix} \delta_{\alpha 3} & \delta_{\alpha 1} - \imath\delta_{\alpha 2} \\ \delta_{\alpha 1} + \imath\delta_{\alpha 2} & -\delta_{\alpha 3} \end{pmatrix}. \tag{3}
$$

The R-matrix (1) satisfies the Yang-Baxter equation

$$
R_{12}(\lambda-\mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda-\mu)\ . \tag{4}
$$

In the present case Yang-Baxter equation reduces to the following matrix equation

$$
\begin{aligned}
\sum_{\alpha,\beta,\gamma=1}^{3} \epsilon_{\alpha\beta\gamma}\, &(W_\beta(\lambda-\mu)W_\gamma(\lambda) - W_\alpha(\lambda-\mu)W_\gamma(\mu) + W_\alpha(\lambda)W_\beta(\mu)\\
&-W_\gamma(\lambda-\mu)W_\beta(\lambda)W_\alpha(\mu))\,\sigma^\alpha\otimes\sigma^\beta\otimes\sigma^\gamma = 0,
\end{aligned} \tag{5}
$$

or to the six scalar equations

$$
\begin{aligned}
&W_2(\lambda-\mu)W_3(\lambda) - W_1(\lambda-\mu)W_3(\mu) + W_1(\lambda)W_2(\mu) - W_3(\lambda-\mu)W_2(\lambda)W_1(\mu) = 0,\\
&W_3(\lambda-\mu)W_1(\lambda) - W_2(\lambda-\mu)W_1(\mu) + W_2(\lambda)W_3(\mu) - W_1(\lambda-\mu)W_3(\lambda)W_2(\mu) = 0,\\
&W_1(\lambda-\mu)W_2(\lambda) - W_3(\lambda-\mu)W_2(\mu) + W_3(\lambda)W_1(\mu) - W_2(\lambda-\mu)W_1(\lambda)W_3(\mu) = 0,\\
&W_3(\lambda-\mu)W_2(\lambda) - W_1(\lambda-\mu)W_2(\mu) + W_1(\lambda)W_3(\mu) - W_2(\lambda-\mu)W_3(\lambda)W_1(\mu) = 0,\\
&W_1(\lambda-\mu)W_3(\lambda) - W_2(\lambda-\mu)W_3(\mu) + W_2(\lambda)W_1(\mu) - W_3(\lambda-\mu)W_1(\lambda)W_2(\mu) = 0,\\
&W_2(\lambda-\mu)W_1(\lambda) - W_3(\lambda-\mu)W_1(\mu) + W_3(\lambda)W_2(\mu) - W_1(\lambda-\mu)W_2(\lambda)W_3(\mu) = 0,
\end{aligned} \tag{6}
$$

which are consequences of the definition (2) of the functions $W_\alpha(\lambda) \equiv W_\alpha(\lambda,\eta,\kappa)$, $\alpha = 1,2,3$, the identities (92) and the addition theorems of the Jacobi elliptic functions (93) - (95).

In the following, we study some important properties of the R-matrix (1). Evidently, this R-matrix has the parity invariance

$$
R_{21}(\lambda) = R_{12}(\lambda), \tag{7}
$$

temporal invariance

$$
R^t_{12}(\lambda) = R_{12}(\lambda), \tag{8}
$$

and it is regular at $\lambda = 0$,

$$R_{12}(0) = 2\mathcal{P}, \tag{9}$$

here $\mathcal{P}$ is the permutation matrix in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

With the aim of checking the unitarity property we calculate the product

$$R(\lambda)R(-\lambda) = \left(1 + \sum_{\alpha=1}^{3} W_\alpha(\lambda) W_\alpha(-\lambda)\right) \mathbb{1} + \sum_{\alpha=1}^{3} \left(W_\alpha(\lambda) + W_\alpha(-\lambda)\right) \sigma^\alpha \otimes \sigma^\alpha$$
$$- \sum_{\alpha,\beta,\gamma,\rho=1}^{3} \epsilon_{\alpha\beta\gamma}\, \epsilon_{\alpha\beta\rho}\, W_\alpha(\lambda) W_\beta(-\lambda)\, \sigma^\gamma \otimes \sigma^\rho. \tag{10}$$

A straightforward calculation shows that

$$\begin{aligned} W_1(\lambda) + W_1(-\lambda) - W_2(\lambda)W_3(-\lambda) - W_3(\lambda)W_2(-\lambda) = 0,\\ W_2(\lambda) + W_2(-\lambda) - W_3(\lambda)W_1(-\lambda) - W_1(\lambda)W_3(-\lambda) = 0,\\ W_3(\lambda) + W_3(-\lambda) - W_1(\lambda)W_2(-\lambda) - W_2(\lambda)W_1(-\lambda) = 0. \end{aligned} \tag{11}$$

Therefore the R-matrix (1) has the unitarity property

$$R(\lambda)R(-\lambda) = \rho(\lambda, \eta, \kappa)\ \mathbb{1} \tag{12}$$

where the function $\rho(\lambda, \eta, \kappa)$ is given by

$$\rho(\lambda, \eta, \kappa) = \left(1 + \sum_{\alpha=1}^{3} W_\alpha(\lambda) W_\alpha(-\lambda)\right) = 4 \frac{\mathrm{sn}^2(\eta, \kappa)}{\mathrm{sn}^2(2\eta, \kappa)}\ \frac{\mathrm{sn}^2(\lambda, \kappa) - \mathrm{sn}^2(2\eta, \kappa)}{\mathrm{sn}^2(\lambda, \kappa) - \mathrm{sn}^2(\eta, \kappa)}. \tag{13}$$

The R-matrix (1) has the crossing symmetry

$$R(\lambda) = \left(\sigma^2 \otimes \mathbb{1}\right) R^{t_1}(-\lambda - 2\eta) \left(\sigma^2 \otimes \mathbb{1}\right), \tag{14}$$

were t_1 denotes the transpositions in the first space of the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$. Consequently, the crossing unitarity property holds

$$R^{t_1}(\lambda) R^{t_1}(\lambda - 4\eta) = \rho(\lambda + 2\eta, \eta, \kappa)\ \mathbb{1}. \tag{15}$$

In what follows we study an inhomogeneous XYZ spin chain with N sites, with the local space V_m, that is the $2s + 1$ dimensional spin s representation space of the Sklyanin algebra and inhomogeneous parameter α_j.

$$\mathcal{H} = \bigotimes_{m=1}^{N} V_m.$$

Following [50] we introduce the Lax operator

$$\mathbb{L}_{0q}(\lambda) = \mathbb{1} \otimes S^0 + \sum_{\alpha=1}^{3} W_\alpha(\lambda,\eta,\kappa)\ \sigma^\alpha \otimes S^\alpha, \tag{16}$$
$$= \begin{pmatrix} S^0 + W_3(\lambda)S^3 & W_1(\lambda)S^1 - \imath W_2(\lambda)S^2 \\ W_1(\lambda)S^1 + \imath W_2(\lambda)S^2 & S^0 - W_3(\lambda)S^3 \end{pmatrix},$$

were S^0, S^1, S^2, S^3 are the generators of the Sklyanin algebra $U_{\tau,\eta}(sl(2))$ [50]. The generators of the Sklyanin algebra satisfy the following relations

$$\begin{aligned} \left[S^1, S^2\right] &= \imath \left(S^0 S^3 + S^3 S^0\right), \\ \left[S^2, S^3\right] &= \imath \left(S^0 S^1 + S^1 S^0\right), \\ \left[S^3, S^1\right] &= \imath \left(S^0 S^2 + S^2 S^0\right), \\ \left[S^0, S^1\right] &= \imath\, J_{23} \left(S^2 S^3 + S^3 S^2\right), \\ \left[S^0, S^2\right] &= \imath\, J_{31} \left(S^3 S^1 + S^1 S^3\right), \\ \left[S^0, S^3\right] &= \imath\, J_{12} \left(S^1 S^2 + S^2 S^1\right), \end{aligned} \tag{17}$$

where

$$J_{23} = \frac{W_2(\lambda-\mu)W_3(\lambda)W_2(\mu) - W_3(\lambda-\mu)W_2(\lambda)W_3(\mu)}{W_1(\lambda) - W_1(\lambda-\mu)W_1(\mu)}, \tag{18}$$

$$J_{23} = \frac{W_3(\lambda-\mu)W_2(\mu)W_3(\lambda) - W_2(\lambda-\mu)W_3(\mu)W_2(\lambda)}{W_1(\mu) - W_1(\lambda-\mu)W_1(\lambda)}, \tag{19}$$

$$J_{31} = \frac{W_3(\lambda-\mu)W_1(\lambda)W_3(\mu) - W_1(\lambda-\mu)W_3(\lambda)W_1(\mu)}{W_2(\lambda) - W_2(\lambda-\mu)W_2(\mu)}, \tag{20}$$

$$J_{31} = \frac{W_1(\lambda-\mu)W_3(\mu)W_1(\lambda) - W_3(\lambda-\mu)W_1(\mu)W_3(\lambda)}{W_2(\mu) - W_2(\lambda-\mu)W_2(\lambda)}, \tag{21}$$

$$J_{12} = \frac{W_1(\lambda-\mu)W_2(\lambda)W_1(\mu) - W_2(\lambda-\mu)W_1(\lambda)W_2(\mu)}{W_3(\lambda) - W_3(\lambda-\mu)W_3(\mu)}, \tag{22}$$

$$J_{12} = \frac{W_2(\lambda-\mu)W_1(\mu)W_2(\lambda) - W_1(\lambda-\mu)W_2(\mu)W_1(\lambda)}{W_3(\mu) - W_3(\lambda-\mu)W_3(\lambda)}. \tag{23}$$

Actually, the quantities J_{12}, J_{23} and J_{31} do not depend on the spectral parameter and are given by [50, 51]

$$J_{12} = \frac{W_1^2(\lambda) - W_2^2(\lambda)}{W_3^2(\lambda) - 1} = (1-\kappa^2)\ \frac{\mathrm{sn}^2(\eta,\kappa)}{\mathrm{cn}^2(\eta,\kappa)\mathrm{dn}^2(\eta,\kappa)}, \tag{24}$$

$$J_{23} = \frac{W_2^2(\lambda) - W_3^2(\lambda)}{W_1^2(\lambda) - 1} = \kappa^2\ \frac{\mathrm{sn}^2(\eta,\kappa)\mathrm{cn}^2(\eta,\kappa)}{\mathrm{dn}^2(\eta,\kappa)}, \tag{25}$$

$$J_{31} = \frac{W_3^2(\lambda) - W_1^2(\lambda)}{W_2^2(\lambda) - 1} = -\frac{\mathrm{sn}^2(\eta,\kappa)\mathrm{dn}^2(\eta,\kappa)}{\mathrm{cn}^2(\eta,\kappa)}. \tag{26}$$

A straightforward calculation shows that

$$J_{12} + J_{23} + J_{31} + J_{12}J_{23}J_{31} = 0. \tag{27}$$

Therefore

$$J_{\alpha\beta} = -\frac{J_\alpha - J_\beta}{J_\gamma}, \tag{28}$$

with

$$J_1 : J_2 : J_3 = \frac{\mathrm{cn}(2\eta,\kappa)}{\mathrm{cn}^2(\eta,\kappa)} : \frac{\mathrm{dn}(2\eta,\kappa)}{\mathrm{dn}^2(\eta,\kappa)} : 1. \tag{29}$$

Also, it is important to notice that [51]

$$\kappa = \frac{J_{12}+1}{J_{12}+J_{23}}\sqrt{-J_{23}J_{31}}, \qquad \mathrm{sn}^2(\eta,\kappa) = \frac{J_{12}+J_{23}}{J_{12}+1}. \tag{30}$$

The quadratic Casimir elements of the Sklyanin algebra are given by

$$C_0 = (S^0)^2 + \sum_{\alpha=1}^{3}(S^\alpha)^2, \tag{31}$$

$$C_2 = \sum_{\alpha=1}^{3} J_\alpha (S^\alpha)^2. \tag{32}$$

Following [50], it maybe useful to introduce another Casimir element of the Sklyanin algebra

$$C_1 = C_0 - C_2 = (S^0)^2 + \sum_{\alpha=1}^{3}(1-J_\alpha)(S^\alpha)^2. \tag{33}$$

Evidently the two dimensional representation of the Sklyanin algebra is given by the R-matrix (1). Therefore in this case

$$S^0 = \mathbb{1} \quad \text{and} \quad S^\alpha = \sigma^\alpha. \tag{34}$$

In this representation, evidently,

$$C_0 = 4\mathbb{1} \quad \text{and} \quad C_2 = (J_1 + J_2 + J_3)\,\mathbb{1}. \tag{35}$$

The other irreducible representations of the Sklyanin algebra are constructed by the so-called fusion procedure [52, 49, 51, 50]. In particular, the three dimensional representation can be obtained from

$$\begin{aligned} R_{1(23)}(\lambda) &= P_{23}^{+} R_{1,23}(\lambda) P_{23}^{+} \\ &= P_{23}^{+} R_{12}(\lambda+\eta) R_{13}(\lambda-\eta) P_{23}^{+}, \end{aligned} \tag{36}$$

where

$$P_{23}^{+} = \mathbb{1} - P_{23}^{-} = \mathbb{1} - \frac{1}{4} R_{23}(-2\eta) \,. \tag{37}$$

It follows that, with a suitable choice of bases, the generators of the Sklyanin algebra are represented by the following set of matrices [50]

$$S^0 = \begin{pmatrix} J_3 & 0 & J_1 - J_2 \\ 0 & J_1 + J_2 - J_3 & 0 \\ J_1 - J_2 & 0 & J_3 \end{pmatrix}, \tag{38}$$

$$S^1 = \sqrt{2J_2J_3} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{39}$$

$$S^2 = \sqrt{2J_3J_1} \begin{pmatrix} 0 & -\imath & 0 \\ \imath & 0 & -\imath \\ 0 & \imath & 0 \end{pmatrix}, \tag{40}$$

$$S^3 = 2\sqrt{J_1J_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{41}$$

Straightforward substitution of formulae (38) - (41) into (31) and (32) yields

$$C_0 = (J_1 + J_2 + J_3)^2 \, \mathbb{1}, \qquad C_2 = (8J_1J_2J_3) \, \mathbb{1}. \tag{42}$$

The commutation relations of the generators of the Sklyanin algebra (17) guarantee the RLL-relations for the XYZ Lax operator (16)

$$R_{12}(\lambda - \mu)\mathbb{L}_{1q}(\lambda)\mathbb{L}_{2q}(\mu) = \mathbb{L}_{2q}(\mu)\mathbb{L}_{1q}(\lambda)R_{12}(\lambda - \mu). \tag{43}$$

The RLL-relations admit a central element

$$\mathbb{D}\left[\mathbb{L}(\lambda)\right] = \mathrm{tr}_{00'} P_{00'}^{-} \mathbb{L}_{0q}(\lambda - \eta)\mathbb{L}_{0'q}(\lambda + \eta), \tag{44}$$

where

$$P_{00'}^{-} = \frac{\mathbb{1} - \mathcal{P}_{00'}}{2} = \frac{1}{4} R_{00'}(-2\eta) \,. \tag{45}$$

A straightforward calculation shows that

$$\Big[\mathbb{D}\big[\mathbb{L}(\mu)\big] \, , \, \mathbb{L}(\nu)\Big] = 0 \; . \tag{46}$$

Substituting (16) into (44) yields

$$\mathbb{D}\left[\mathbb{L}(\lambda)\right] = (S^0)^2 - \sum_{\alpha=1}^{3} W_\alpha(\lambda - \eta)W_\alpha(\lambda + \eta)(S^\alpha)^2. \tag{47}$$

In particular, in the case of the R-matrix, its quantum determinant is given by

$$\mathbb{D}\left[R(\lambda)\right] = \left(1 - \sum_{\alpha=1}^{3} W_\alpha(\lambda-\eta)W_\alpha(\lambda+\eta)\right)\mathbb{1} = \rho(\lambda+\eta,\eta,\kappa)\mathbb{1}\ . \tag{48}$$

In general, the central $\mathbb{D}\left[\mathbb{L}(\lambda)\right]$ can be expressed in terms of the Casimir elements of the Sklyanin algebra using (28), (19), (21) and (23),

$$\mathbb{D}\left[\mathbb{L}(\lambda)\right] = C_0 - \frac{1+W_3(\lambda-\eta)W_3(\lambda+\eta)}{J_3}\,C_2\ . \tag{49}$$

The quantum determinant $\mathbb{D}\left[\mathbb{L}(\lambda)\right]$ can be written in terms of C_0 and C_1 as follows

$$\begin{aligned}\mathbb{D}\left[\mathbb{L}(\lambda)\right] = &\left(1 - \frac{1+W_3(\lambda-\eta)W_3(\lambda+\eta)}{J_3}\right)C_0\\ &+\left(\frac{1+W_3(\lambda-\eta)W_3(\lambda+\eta)}{J_3}\right)C_1.\end{aligned} \tag{50}$$

The Lax operator (16) satisfies the following identity

$$\mathbb{L}_{0q}(\lambda)\,\sigma_0^2\mathbb{L}_{0q}^{t_0}(\lambda-2\eta)\sigma_0^2 = \mathbb{1}\otimes\mathbb{D}\left[\mathbb{L}(\lambda-\eta)\right], \tag{51}$$

were t_0 denotes the transpositions in the auxiliary space $\mathbb{C}^2$.

The so-called monodromy matrix

$$T(\lambda) = \mathbb{L}_{0N}(\lambda-\alpha_N)\cdots\mathbb{L}_{01}(\lambda-\alpha_1) \tag{52}$$

is used to describe the system. Notice that $T(\lambda)$ is a two-by-two matrix in the auxiliary space $V_0 = \mathbb{C}^2$, whose entries are operators acting in $\mathcal{H}$

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda)\\ C(\lambda) & D(\lambda)\end{pmatrix}. \tag{53}$$

From RLL-relations it follows that the monodromy matrix satisfies the RTT-relations

$$R_{00'}(\lambda-\mu)T_0(\lambda)T_{0'}(\mu) = T_{0'}(\mu)T_0(\lambda)R_{00'}(\lambda-\mu). \tag{54}$$

The RTT-relations define the commutation relations for the entries of the monodromy matrix.

In the periodic case the modified Algebraic Bethe Ansatz [48, 56] yields the spectrum of the spin-s XYZ Heisenberg Hamiltonian

$$H = -\frac{1}{2}\sum_{m=1}^{N}\left(J_1S_m^1S_{m+1}^1 + J_2S_m^2S_{m+1}^2 + J_3S_m^3S_{m+1}^3\right). \tag{55}$$

A way to introduce non-periodic boundary conditions which are compatible with the integrability of the bulk system, was developed in [27].

The compatibility condition between the bulk and the boundary of the system takes the form of the so-called Reflection Equation. It is written in the following form for the left reflection matrix acting on the space $V_1 = \mathbb{C}^2$ at the first site, $K^-(u) \in \mathrm{End}(\mathbb{C}^2)$

$$R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v) = K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v). \tag{56}$$

The general solution of the reflection equation above, for the R-matrix (16), can be written as follows [53, 54, 55]

$$K^-(u) = \begin{pmatrix} \mathrm{sn}(u+a) - d\,\mathrm{sn}(u-a) & b\,\mathrm{sn}(2u)\,\dfrac{c(1-\tau\,\mathrm{sn}^2(u)) + 1 + \tau\,\mathrm{sn}^2(u)}{1-\tau^2\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(a)} \\ b\,\mathrm{sn}(2u)\,\dfrac{c(1-\tau\,\mathrm{sn}^2(u)) - 1 - \tau\,\mathrm{sn}^2(u)}{1-\tau^2\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(a)} & -\mathrm{sn}(u-a) + d\,\mathrm{sn}(u+a) \end{pmatrix}, \tag{57}$$

here a, b, c, d are arbitrary constants.

Due to the properties of the Yang R-matrix the dual reflection equation can be presented in the following form

$$R_{12}(v-u)K_1^+(u)R_{21}(-u-v-2\omega)K_2^+(v) = K_2^+(v)R_{12}(-u-v-2\omega)K_1^+(u)R_{21}(v-u).$$

One can then verify that the mapping

$$K^+(u) = K^-(-u-\omega)$$

is a bijection between solutions of the reflection equation and the dual reflection equation. After substitution of into the dual reflection equation one gets the reflection equation with shifted arguments.

We use Sklyanin approach to integrable spin chains with non-periodic boundary conditions [27]. The Sklyanin monodromy matrix $\mathcal{T}(\lambda)$ is

$$\mathcal{T}_0(\lambda) = T_0(\lambda)K_0^-(\lambda)\widetilde{T}_0(\lambda). \tag{58}$$

The monodromy matrix $\widetilde{T}_0(\lambda)$ is such that its RTT-relations (54) can be recast as follows

$$\widetilde{T}_{0'}(\mu)R_{00'}(\lambda+\mu)T_0(\lambda) = T_0(\lambda)R_{00'}(\lambda+\mu)\widetilde{T}_{0'}(\mu), \tag{59}$$

$$\widetilde{T}_0(\lambda)\widetilde{T}_{0'}(\mu)R_{00'}(\mu-\lambda) = R_{00'}(\mu-\lambda)\widetilde{T}_{0'}(\mu)\widetilde{T}_0(\lambda). \tag{60}$$

Then, by construction, the exchange relations of the monodromy matrix $\mathcal{T}(\lambda)$ are

$$R_{00'}(\lambda-\mu)\mathcal{T}_0(\lambda)R_{0'0}(\lambda+\mu)\mathcal{T}_{0'}(\mu) = \mathcal{T}_{0'}(\mu)R_{00'}(\lambda+\mu)\mathcal{T}_0(\lambda)R_{0'0}(\lambda-\mu). \tag{61}$$

The Reflection Equation Algebra admits a central element, the so-called Sklyanin determinant [27],

$$\Delta\left[\mathcal{T}(\lambda)\right] = \mathrm{tr}_{00'} P^{-}_{00'} \mathcal{T}_0(\lambda - \eta/2) R_{00'}(2\lambda) \mathcal{T}_{0'}(\lambda + \eta/2). \tag{62}$$

The open chain transfer matrix is given by the trace of the monodromy $\mathcal{T}(\lambda)$ over the auxiliary space V_0 with an extra reflection matrix $K^{+}(\lambda)$,

$$t(\lambda) = \mathrm{tr}_0 \left(K^{+}(\lambda) \mathcal{T}(\lambda) \right). \tag{63}$$

The reflection matrix $K^{+}(\lambda)$ is the corresponding solution of the dual reflection equation.

The commutativity of the transfer matrix for different values of the spectral parameter

$$[t(\lambda), t(\mu)] = 0, \tag{64}$$

is guaranteed by the dual reflection equation and the exchange relations of the monodromy matrix $\mathcal{T}(\lambda)$.

In the spin-$\frac{1}{2}$ case, in the homogeneous limit, this transfer matrix yields the Hamiltonian with the boundary terms of the form:

$$H = \sum_{m=1}^{N-1} H_{m,m+1} + \left(A_{-}\sigma_1^z + B_{-}\sigma_1^{+} + C_{-}\sigma_1^{-} \right) + \left(A_{+}\sigma_N^z + B_{+}\sigma_N^{+} + C_{+}\sigma_N^{-} \right), \tag{65}$$

where the first term corresponds to the expression 55 (here without the periodic boundary condition). The separation of variables method [57, 58] was used to obtain the spectrum of the above transfer matrix by S. Faldella and G. Niccoli [59].

3. Generalized XYZ Gaudin model

The key observation is that the R-matrix (1) admits the following expansion [16]

$$R(\lambda, \eta) = \mathbb{1} + 2\eta r(\lambda) + \mathcal{O}(\eta^2), \tag{66}$$

where the classical r-matrix is given by

$$r(\lambda) = \sum_{\alpha=1}^{3} w_\alpha(\lambda)\ \sigma^\alpha \otimes \sigma^\alpha, \tag{67}$$

where

$$w_1(\lambda) = \frac{\mathrm{cn}(\lambda,\kappa)}{\mathrm{sn}(\lambda,\kappa)}, \quad w_2(\lambda) = \frac{\mathrm{dn}(\lambda,\kappa)}{\mathrm{sn}(\lambda,\kappa)}, \quad w_3(\lambda) = \frac{1}{\mathrm{sn}(\lambda,\kappa)}, \tag{68}$$

the functions $\mathrm{sn}(\lambda, \kappa)$, $\mathrm{cn}(\lambda, \kappa)$, and $\mathrm{dn}(\lambda, \kappa)$ are the usual Jacobi elliptic functions and σ^α, $\alpha = 1, 2, 3$, are the Pauli matrices (3). Evidently, this classical r-matrix has the parity invariance property

$$r_{21}(\lambda) = r_{12}(\lambda), \tag{69}$$

and, due to the fact that $w_\alpha(\lambda)$ (68) are odd function of λ cf. (91), it also has the unitarity property

$$r_{21}(-\lambda) = -r_{12}(\lambda). \tag{70}$$

Notice that in the case of the r-matrix (67) the classical Yang-Baxter equation

$$[r_{13}(\lambda), r_{23}(\mu)] + [r_{12}(\lambda - \mu), r_{13}(\lambda) + r_{23}(\mu)] = 0, \tag{71}$$

reduces to the following three identities

$$\begin{aligned} w_1(\lambda)\ w_2(\mu) &= -w_2(\lambda-\mu)\ w_3(\lambda) + w_1(\lambda-\mu)\ w_3(\mu), \\ w_3(\lambda)\ w_1(\mu) &= -w_1(\lambda-\mu)\ w_2(\lambda) + w_3(\lambda-\mu)\ w_2(\mu), \\ w_2(\lambda)\ w_3(\mu) &= -w_3(\lambda-\mu)\ w_1(\lambda) + w_2(\lambda-\mu)\ w_1(\mu), \end{aligned} \tag{72}$$

which are consequences of the definition (68) of the functions $w_\alpha(\lambda)$, $\alpha = 1, 2, 3$, the identities (92) and the addition theorems of the Jacobi elliptic functions (93) - (95).

The Lax operator of the chain (16) admits the following expansion

$$\mathbb{L}_{0q}(\lambda, \eta) = \mathbb{1} + 2\eta\ \ell_{0q}(\lambda) + \mathcal{O}(\eta^2), \tag{73}$$

where

$$\ell_{0q}(\lambda) = \sum_{\alpha=1}^{3} w_\alpha(\lambda)\ \sigma_0^\alpha \otimes S^\alpha. \tag{74}$$

Therefore the expansion of the monodromy matrix (52) reads

$$T_0(\lambda, \eta) = \mathbb{1} + 2\eta\ L_0(\lambda) + \eta^2\ T_0^{(2)}(\lambda) + \mathcal{O}(\eta^3), \tag{75}$$

where the Gaudin Lax operator is given by

$$L_0(\lambda) = \sum_{m=1}^{N} \ell_{0n}(\lambda - \alpha_m). \tag{76}$$

The RTT-relations (54) imply the so-called Sklyanin linear bracket for the Gaudin Lax operator

$$[L_0(\lambda), L_{0'}(\mu)] = [r_{00'}(\lambda - \mu), L_0(\lambda) + L_{0'}(\mu)]\,, \tag{77}$$

with the above classical r-matrix (67).

It can be shown that the transfer matrix of the chain and the quantum determinant of the monodromy matrix admit the following expansions

$$\begin{aligned} t(\lambda,\eta) &= 1+\eta^2 \ \mathrm{tr}_0 T_0^{(2)}(\lambda) + \mathcal{O}(\eta^3), \\ \mathbb{D}\left[T_0(\lambda,\eta)\right] &= 1+\eta^2 \ \left(\mathrm{tr}_0 T_0^{(2)}(\lambda) + 4 \ \mathrm{tr}_0 L_0^2(\lambda)\right) + \mathcal{O}(\eta^3). \end{aligned} \tag{78}$$

Thus the generating function $\tau(\lambda)$ of the Gaudin Hamiltonians in the elliptic case can be obtain as a difference

$$\mathbb{D}\left[T_0(\lambda,\eta)\right] - t(\lambda,\eta) = 4\eta \ \mathrm{tr}_0 L_0^2(\lambda) + \mathcal{O}(\eta^3), \tag{79}$$

with, as expected,

$$\tau(\lambda) = \mathrm{tr}_0 L_0^2(\lambda). \tag{80}$$

Evidently, $\tau(\lambda)$ commute for different values of the spectral parameter,

$$[\tau(\lambda), \tau(\mu)] = 0. \tag{81}$$

As in the rational and the trigonometric case, the expansion into partial fractions yields the corresponding Gaudin Hamiltonians

$$\tau(\lambda) = \sum_{m=1}^{N} \wp(\lambda-\alpha_m) \ s_m(s_m+1) + \sum_{m=1}^{N} \zeta(\lambda-\alpha_m) \ H_m + H_0, \tag{82}$$

where $\wp$ is the Weierstrass P function, ζ is the zeta function, H_0 is λ independent and

$$H_m = 2\sum_{m\neq n}\sum_{\alpha=1}^{3} w_\alpha(\alpha_n-\alpha_m) \ S_n^\alpha S_m^\beta. \tag{83}$$

Sklyanin and Takebe [16, 17] obtained the spectrum of the generating function both by the modified Algebraic Bethe Ansatz and by the separation of variables method.

In order to define the Gaudin model with boundary terms we consider the following non-unitary classical r-matrix [44, 47]

$$r_{00'}^K(\lambda,\mu) = r_{00'}(\lambda-\mu) - K_{0'}(\nu) r_{00'}(\lambda+\mu) K_{0'}^{-1}(\mu), \tag{84}$$

where

$$K_0(\lambda) \equiv K_0^-(\lambda). \tag{85}$$

It is straightforward to check that this r-matrix satisfies the classical Yang-Baxter equation [44, 47]

$$[r_{32}^K(\lambda_3,\lambda_2), r_{13}^K(\lambda_1,\lambda_3)] + [r_{12}^K(\lambda_1,\lambda_2), r_{13}^K(\lambda_1,\lambda_3) + r_{23}^K(\lambda_2,\lambda_3)] = 0. \tag{86}$$

The corresponding Lax operator is given by

$$\mathcal{L}_0(\lambda) = \sum_{m=1}^{N} \left(\ell_{0m}(\lambda - \alpha_m) + K_0(\lambda) \ell_{0m}(\lambda + \alpha_m) K_0^{-1}(\lambda) \right). \tag{87}$$

Evidently, it satisfies the following linear bracket relations

$$[\mathcal{L}_0(\lambda), \mathcal{L}_{0'}(\mu)] = \left[r_{00'}^K(\lambda, \mu), \mathcal{L}_0(\lambda) \right] - \left[r_{0'0}^K(\mu, \lambda), \mathcal{L}_{0'}(\mu) \right]. \tag{88}$$

By definition this linear bracket is obviously anti-symmetric. It obeys the Jacobi identity since the r-matrix $r_{00'}^K(\lambda, \mu)$ satisfies the classical Yang-Baxter equation.

The generating function $\tau(\lambda)$ of the Gaudin Hamiltonians with boundary terms is given by

$$\tau(\lambda) = \mathrm{tr}_0\, \mathcal{L}_0^2(\lambda). \tag{89}$$

The generating function for different values of the spectral parameter obviously commute,

$$[\tau(\lambda), \tau(\mu)] = 0. \tag{90}$$

Thus, defining the family of the Gaudin Hamiltonians in involution.

In this way, we have achieved the main objective of this work. Our next step will be to solve explicitly the model using either the modified algebraic Bethe ansatz [48], which was successfully used in the periodic case [16], or the separation of variables method [57, 58], used to solve the periodic elliptic Gaudin model [17] as well as the non-periodic elliptic chain [59].

A Jacobi elliptic functions

In this appendix we briefly review some basic identities which follow from the definition of the Jacobi elliptic functions. In particular,

$$\mathrm{sn}(-\lambda, \kappa) = -\mathrm{sn}(\lambda, \kappa), \quad \mathrm{cn}(-\lambda, \kappa) = \mathrm{cn}(\lambda, \kappa), \quad \mathrm{dn}(-\lambda, \kappa) = \mathrm{dn}(\lambda, \kappa), \tag{91}$$

$$\mathrm{sn}^2(\lambda, \kappa) + \mathrm{cn}^2(\lambda, \kappa) = 1, \quad \mathrm{dn}^2(\lambda, \kappa) + \kappa^2 \mathrm{sn}^2(\lambda, \kappa) = 1. \tag{92}$$

The addition theorems for the Jacobi elliptic functions read

$$\mathrm{sn}(\lambda + \mu, \kappa) = \frac{\mathrm{sn}(\lambda, \kappa)\mathrm{cn}(\mu, \kappa)\mathrm{dn}(\mu, \kappa) + \mathrm{cn}(\lambda, \kappa)\mathrm{dn}(\lambda, \kappa)\mathrm{sn}(\mu, \kappa)}{1 - \kappa^2 \mathrm{sn}^2(\lambda, \kappa)\mathrm{sn}^2(\mu, \kappa)}, \tag{93}$$

$$\mathrm{cn}(\lambda + \mu, \kappa) = \frac{\mathrm{cn}(\lambda, \kappa)\mathrm{cn}(\mu, \kappa) - \mathrm{sn}(\lambda, \kappa)\mathrm{dn}(\lambda, \kappa)\mathrm{sn}(\mu, \kappa)\mathrm{dn}(\mu, \kappa)}{1 - \kappa^2 \mathrm{sn}^2(\lambda, \kappa)\mathrm{sn}^2(\mu, \kappa)}, \tag{94}$$

$$\mathrm{dn}(\lambda + \mu, \kappa) = \frac{\mathrm{dn}(\lambda, \kappa)\mathrm{dn}(\mu, \kappa) - \kappa^2 \mathrm{sn}(\lambda, \kappa)\mathrm{cn}(\lambda, \kappa)\mathrm{sn}(\mu, \kappa)\mathrm{cn}(\mu, \kappa)}{1 - \kappa^2 \mathrm{sn}^2(\lambda, \kappa)\mathrm{sn}^2(\mu, \kappa)}. \tag{95}$$

Some other important identities follow from the formulae above

$$\operatorname{sn}(\lambda+\mu,\kappa)\operatorname{sn}(\lambda-\mu,\kappa)=\frac{\operatorname{sn}^2(\lambda,\kappa)-\operatorname{sn}^2(\mu,\kappa)}{1-\kappa^2\operatorname{sn}^2(\lambda,\kappa)\operatorname{sn}^2(\mu,\kappa)}, \tag{96}$$

$$\operatorname{cn}(\lambda+\mu,\kappa)\operatorname{cn}(\lambda-\mu,\kappa)=\frac{\operatorname{cn}^2(\mu,\kappa)-\operatorname{sn}^2(\lambda,\kappa)\operatorname{dn}^2(\mu,\kappa)}{1-\kappa^2\operatorname{sn}^2(\lambda,\kappa)\operatorname{sn}^2(\mu,\kappa)}, \tag{97}$$

$$\operatorname{dn}(\lambda+\mu,\kappa)\operatorname{dn}(\lambda-\mu,\kappa)=\frac{\operatorname{dn}^2(\mu,\kappa)-\kappa^2\operatorname{sn}^2(\lambda,\kappa)\operatorname{cn}^2(\mu,\kappa)}{1-\kappa^2\operatorname{sn}^2(\lambda,\kappa)\operatorname{sn}^2(\mu,\kappa)}. \tag{98}$$

References

[1] M. Gaudin, *Diagonalisation d'une classe d'hamiltoniens de spin*, J. Physique 37 (1976) 1087–1098.

[2] M. Gaudin, *La fonction d'onde de Bethe*, chapter 13 Masson, Paris, 1983.

[3] M. Gaudin, *The Bethe Wavefunction*, Cambridge University Press, 2014.

[4] E. K. Sklyanin, *Separation of variables in the Gaudin model*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 164 (1987) 151–169; translation in J. Soviet Math. 47 (1989) 2473–2488.

[5] A. A. Belavin and V. G. Drinfeld. *Solutions of the classical Yang-Baxter equation for simple Lie algebras* (in Russian), Funktsional. Anal. i Prilozhen. 16 (1982), no. 3, 1–29; translation in Funct. Anal. Appl. 16 (1982) no. 3, 159-180.

[6] M. A. Semenov-Tian-Shansky, *Quantum and classical integrable systems*, in Integrability of Nonlinear Systems, Lecture Notes in Physics Volume 495 (1997) 314-377.

[7] B. Jurčo, *Classical Yang-Baxter equations and quantum integrable systems*, J. Math. Phys. Volume 30 (1989) 1289–1293.

[8] B. Jurčo, *Classical Yang-Baxter equations and quantum integrable systems (Gaudin models)*, in Quantum groups (Clausthal, 1989), Lecture Notes in Phys. Volume 370 (1990) 219–227.

[9] B. Feigin, E. Frenkel, and N. Reshetikhin, *Gaudin model, Bethe ansatz and correlation functions at the critical level*, Commun. Math. Phys. 166 (1994) 27–62.

[10] N. Reshetikhin and A. Varchenko, *Quasiclassical asymptotics of solutions to the KZ equations*, in Geometry, Topology and Physics, Conf. Proc. Lecture Notes Geom., pages 293–273. Internat. Press, Cambridge, MA, 1995.

[11] F. Wagner and A. J. Macfarlane, *Solvable Gaudin models for higher rank symplectic algebras. Quantum groups and integrable systems (Prague, 2000)* Czechoslovak J. Phys. 50 (2000) 1371–1377.

[12] T. Brzezinski and A. J. Macfarlane, *On integrable models related to the osp(1,2) Gaudin algebra*, J. Math. Phys. 35 (1994), no. 7, 3261–3272.

[13] P. P. Kulish and N. Manojlović, *Bethe vectors of the* $osp(1|2)$ *Gaudin model*, Letters in Mathematical Physics **Vol. 55** (2001) 77-95.

[14] P. P. Kulish and N. Manojlović, *Creation operators and Bethe vectors of the* $osp(1|2)$ *Gaudin model*, J. Math. Phys. 42 no. 10 (2001) 4757–4778.

[15] P. P. Kulish and N. Manojlović, *Trigonometric* $osp(1|2)$ *Gaudin model*, J. Math.Phys. 44 no. 2 (2003) 676–700.

[16] E. K. Sklyanin and T. Takebe, *Algebraic Bethe ansatz for the XYZ Gaudin model*, Physics Letters A 219 (1996) 217–225.

[17] E. K. Sklyanin and T. Takebe, *Separation of Variables in the Elliptic Gaudin Model*, Commun. Math. Phys. 204 (1999) 17– 38.

[18] K. Hikami, P. P. Kulish and M. Wadati, *Integrable Spin Systems with Long-Range Interaction*, Chaos, Solitons & Fractals Vol. 2 No. 5 (1992) 543–550.

[19] K. Hikami, P. P. Kulish and M. Wadati, *Construction of Integrable Spin Systems with Long-Range Interaction*, J. Phys. Soc. Japan Vol. 61 No. 9 (1992) 3071–3076.

[20] K. Hikami, *Gaudin magnet with boundary and generalized Knizhnik-Zamolodchikov equation*, J. Phys. A Math. Gen. **28** (1995) 4997–5007.

[21] W. L. Yang, R. Sasaki and Y. Z. Zhang, $\mathbb{Z}_n$ *elliptic Gaudin model with open boundaries*, JHEP 09 (2004) 046.

[22] W. L. Yang, R. Sasaki and Y. Z. Zhang, A_{n-1} *Gaudin model with open boundaries*, Nuclear Physics B 729 (2005) 594–610.

[23] K. Hao, W.-L. Yang, H. Fan, S. Y. Liu, K. Wu, Z. Y. Yang and Y. Z. Zhang, *Determinant representations for scalar products of the XXZ Gaudin model with general boundary terms*, Nuclear Physics **B 862** (2012) 835–849.

[24] A. Lima-Santos, *The* $sl(2|1)^{(2)}$ *Gaudin magnet with diagonal boundary terms*, J. Stat. Mech. (2009) P07025.

[25] E. K. Sklyanin, *Boundary conditions for integrable equations*, (Russian) Funktsional. Anal. i Prilozhen. 21 (1987) 86–87; translation in Functional Analysis and Its Applications Volume 21, Issue 2 (1987) 164–166.

[26] E. K. Sklyanin, *Boundary conditions for integrable systems*, in the Proceedings of the VIIIth international congress on mathematical physics (Marseille, 1986), World Sci. Publishing, Singapore, (1987) 402–408.

[27] E. K. Sklyanin, *Boundary conditions for integrable quantum systems*, J. Phys. A: Math. Gen. **21** (1988) 2375–2389.

[28] T. Skrypnyk, *Non-skew-symmetric classical r-matrix, algebraic Bethe ansatz, and Bardeen-Cooper-Schrieffer-type integrable systems*, J. Math. Phys. 50 (2009) 033540, 28 pages.

[29] T. Skrypnyk, *"Z2-graded" Gaudin models and analytical Bethe ansatz*, Nuclear Physics B 870 (2013), no. 3, 495–529.

[30] N. Cirilo António, N. Manojlović and Z. Nagy, *Trigonometric* $s\ell(2)$ *Gaudin model with boundary terms*, Reviews in Mathematical Physics **Vol. 25** No. 10 (2013) 1343004 (14 pages).

[31] J. Cao, H. Lin, K. Shi and Y. Wang, *Exact solutions and elementary excitations in the XXZ spin chain with unparallel boundary fields*, Nucl. Phys. B 663 (2003) 487–519.

[32] R. I. Nepomechie, *Bethe ansatz solution of the open XXZ chain with nondiagonal boundary terms*, J. Phys. A 37 (2004), no. 2, 433–440.

[33] D. Arnaudon, A. Doikou, L. Frappat, E. Ragoucy and N. Crampé, *Analytical Bethe ansatz in* $gl(N)$ *spin chains*, Czechoslovak J. Phys. 56 (2006), no. 2, 141–148.

[34] C. S. Melo, G. A. P. Ribeiro and M. J. Martins, *Bethe ansatz for the* $XXX-S$ *chain with non-diagonal open boundaries*, Nuclear Phys. B 711, no. 3 (2005) 565–603.

[35] J. Cao, W.-L. Yang, K. Shi and Y. Wang, *Off-diagonal Bethe ansatz solution of the XXX spin chain with arbitrary boundary conditions*, Nuclear Physics B 875 (2013) 152–165.

[36] J. Cao, W.-L. Yang, K. Shi and Y. Wang, *Off-diagonal Bethe ansatz solutions of the anisotropic spin-1/2 chains with arbitrary boundary fields*, Nuclear Physics B 877 (2013) 152–175.

[37] E. Ragoucy, *Coordinate Bethe ansätze for non-diagonal boundaries*, Rev. Math. Phys. 25 (2013), no. 10, 1343007.

[38] S. Belliard, N. Crampé and E. Ragoucy, *Algebraic Bethe ansatz for open XXX model with triangular boundary matrices*, Lett. Math. Phys. 103 No. 5 (2013) 493–506.

[39] S. Belliard and N. Crampé *Heisenberg XXX model with general boundaries: eigenvectors from algebraic Bethe ansatz*, SIGMA Symmetry Integrability Geom. Methods Appl. 9 (2013), Paper 072, 12 pp.

[40] R. A. Pimenta and A. Lima-Santos, *Algebraic Bethe ansatz for the six vertex model with upper triangular K-matrices*, J. Phys. A 46 No. 45 (2013) 455002, 13 pp.

[41] N. Cirilo António, N. Manojlović and I. Salom, *Algebraic Bethe ansatz for the XXX chain with triangular boundaries and Gaudin model*, Nuclear Physics B 889 (2014) 87-108.

[42] K. Hao, J. Cao, T. Yang and W.-L. Yang, *Exact solution of the XXX Gaudin model with the generic open boundaries*, `arXiv:1408.3012`.

[43] N. Manojlović and I. Salom, *Algebraic Bethe ansatz for the XXZ Heisenberg spin chain with triangular boundaries and the corresponding Gaudin model*, Nuclear Physics **B 923** (2017) 73-106; `arXiv:1705.02235`.

[44] N. Cirilo António, N. Manojlović, E. Ragoucy and I. Salom, *Algebraic Bethe ansatz for the $s\ell(2)$ Gaudin model with boundary*, Nuclear Physics **B 893** (2015) 305-331; `arXiv:1412.1396`.

[45] I. Salom and N. Manojlović, *Creation operators of the non-periodic $sl(2)$ Gaudin model*, Proceedings of the 8th Mathematical Physics meeting: Summer School and Conference on Modern Mathematical Physics, 24 - 31 August 2014, Belgrade, Serbia, SFIN **XXVIII** Series A: Conferences No. A1, ISBN: 978-86-82441-43-4, (2015) 149–155.

[46] I. Salom, N. Manojlović and N. Cirilo António, *Generalized $s\ell(2)$ Gaudin algebra and corresponding Knizhnik-Zamolodchikov equation*, Nuclear Physics **B 939** (2019) 358-371.

[47] N. Manojlović and I. Salom, *Algebraic Bethe ansatz for the trigonometric $s\ell(2)$ Gaudin model with triangular boundary*, Symmetry 12 (2020) 352; `arXiv:1709.06419`.

[48] L. A. Takhtajan and L. D. Faddeev, *The quantum method for the inverse problem and the XYZ Heisenberg model*, (in Russian) Uspekhi Mat. Nauk **34** No. 5 (1979) 13–63; translation in Russian Math. Surveys **34** No.5 (1979) 11–68.

[49] P. P. Kulish and E. K. Sklyanin, *Quantum spectral transform method. Recent developments*, Lect. Notes Phys. **151** (1982), 61–119.

[50] E. K. Sklyanin, *Some algebraic structures connected with the Yang-Baxter equation*, Funkts. Analiz i ego Prilozh. **16** No. 4 (1982) 27–34; translation in Functional Analysis and Applications **17** (1983) 263–270.

[51] E. K. Sklyanin, *Some algebraic structures connected with the Yang-Baxter equation. Representations of quantum algebras*, Funkts. Analiz i ego Prilozh. **17** No. 4 (1983) 34–48; translation in Functional Analysis and Applications **17** (1984) 273–284.

[52] P. P. Kulish, N. Yu. Reshetikhin and E. K. Sklyanin, *Yang-Baxter equation and representation theory I*, Letters in Mathematical Physics **5** (1981) 393–403.

[53] H. J. de Vega and A. Gonzalez-Ruiz, *Boundary K matrices for the XYZ, XXZ and XXX spin chains*, J. Phys. A **27** (1994) 6129–6137.

[54] T. Inami and H. Konno, *Integrable XYZ spin chain with boundaries*, J. Phys. A **27** (1994) L913-L918.

[55] Y. Komori and K. Hikami, *Elliptic K-matrix associated with Belavin's symmetric R-matrix*, Nucl. Phys. B **494** (1997) 687–701.

[56] T. Takebe, *Generalized Bethe ansatz with the general spin representations of the Sklyanin algebra*, J. Phys. A: Math. Gen. 25 (1992) 1071-1083.

[57] E. K. Sklyanin, *Separation of variables - new trends*, Prog. Theor. Phys. Suppl. **118** (1995) 35–60.

[58] J. M. Maillet and G. Niccoli, *On Separation of Variables for Reflection Algebras*, J. Stat. Mech. **1909**, no. 9, (2019) 094020.

[59] S. Faldella and G. Niccoli, *SOV approach for integrable quantum models associated with general representations on spin-1/2 chains of the 8-vertex reflection algebra*, J. Phys. A **47** (2014) 115202.