The spin 1 XXZ Gaudin model with boundary*

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Abstract

We obtain the non-unitary classical r-matrix of the spin 1 trigonometric Gaudin model with boundary terms. Starting from the R-matrix and corresponding K-matrix of the spin $\frac{1}{2}$ XXZ Heisenberg chain the so-called fusion procedure yields the R and K matrices of the spin 1 XXZ Heisenberg chain. We demonstrate that the corresponding classical r and K matrices satisfy, respectively, the classical Yang-Baxter equation and the classical reflection equation. Consequently, we show that the relevant non-unitary classical r-matrix satisfies the generalized classical Yang-Baxter equation and therefore defines the non-periodic spin 1 trigonometric Gaudin model.

1. Introduction

An approach to study periodic Gaudin models [1, 2, 3] is based on the unitary classical r-matrices [4, 5, 6, 7] and the corresponding Sklyanin linear

* N.M. gratefully acknowledges useful discussions with Eric Ragoucy. I.S. was supported in part by the Serbian Ministry of Education, Science and Technological Development under grant number ON 171031.

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bracket [5, 6, 7, 8, 9, 10, 11]. The algebra generated by the entries of the relevant Lax operators can be used to completely solve the system through the algebraic Bethe ansatz, or the separation of variable method [5, 6, 7, 8, 9, 10, 11].

Similar approach to the non-periodic Gaudin models, based on the non-unitary classical r-matrices and corresponding linear bracket, enabled full implementation of the algebraic Bethe ansatz in the spin $\frac{1}{2}$ case [12, 13, 14, 15, 16, 17, 18, 19]. In this paper we initiate the study of the spin 1 non-periodic Gaudin models by deriving the relevant non-unitary classical r-matrix. This classical r-matrix is a solution to the generalized classical Yang-Baxter equation. An important advantage of this approach is that the generalized classical Yang-Baxter equation is equivalent to both the classical Yang-Baxter equation and classical reflection equation [15]. To obtain this non-unitary r-matrix we use the fusion procedure starting from the quantum R-matrix and the reflection K-matrix of the spin $\frac{1}{2}$ XXZ Heisenberg chain. The classical r-matrix we obtain defines completely the non-periodic spin 1 Gaudin model.

This paper is organized as follows. In the section 2. we describe briefly the fusion procedure as applied to the R and K matrices of the spin $\frac{1}{2}$ XXZ Heisenberg chain. The results of the second section are then presented and analysed in the section 3.. The quasi-classical limit which leads to the spin 1 non-unitary r-matrix is presented in the section 4..

2. Spin $\frac{1}{2}$ XXZ Heisenberg chain

In our study of the spin $\frac{1}{2}$ XXZ Heisenberg spin chain [16] the starting point is always the R-matrix [20, 21, 22, 23]

$$R(\lambda,\eta)=\begin{pmatrix} \sinh(\lambda+\eta) & 0 & 0 & 0 \\ 0 & \sinh(\lambda) & \sinh(\eta) & 0 \\ 0 & \sinh(\eta) & \sinh(\lambda) & 0 \\ 0 & 0 & 0 & \sinh(\lambda+\eta) \end{pmatrix}. \tag{1}$$

This R-matrix satisfies the Yang-Baxter equation [24, 22, 23, 20, 21] in the space $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$

$$R_{12}(\lambda-\mu)R_{13}(\lambda)R_{23}(\mu)=R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda-\mu), \tag{2}$$

and it also has other relevant properties such as

$U(1)$ symmetry	$\left[\sigma_1^3+\sigma_2^3, R_{12}(\lambda)\right]=0;$
unitarity	$R_{12}(\lambda)R_{21}(-\lambda)=\sinh(\eta-\lambda)\sinh(\eta+\lambda)\mathbb{1};$
parity invariance	$R_{21}(\lambda)=R_{12}(\lambda);$
temporal invariance	$R_{12}^t(\lambda)=R_{12}(\lambda);$
crossing symmetry	$R(\lambda)=\mathcal{J}_1R^{t_2}(-\lambda-\eta)\mathcal{J}_1,$

where t_2 denotes the transpose in the second space and the two-by-two matrix $\mathcal{J}$ is proportional to the Pauli matrix σ^2, i.e. $\mathcal{J} = \imath\sigma^2$.

A way to introduce non-periodic boundary conditions which are compatible with the integrability of the bulk model, was developed in [25]. Boundary conditions on the left and right sites of the chain are encoded in the left and right reflection matrices K^- and K^+. The compatibility condition between the bulk and the boundary of the system takes the form of the so-called reflection equation. It is written in the following form for the left reflection matrix acting on the space $\mathbb{C}^2$ at the first site $K^-(\lambda) \in \mathrm{End}(\mathbb{C}^2)$

$$R_{12}(\lambda-\mu)K_1^-(\lambda)R_{21}(\lambda+\mu)K_2^-(\mu) = K_2^-(\mu)R_{12}(\lambda+\mu)K_1^-(\lambda)R_{21}(\lambda-\mu). \tag{3}$$

Due to the properties of the R-matrix (1) the dual reflection equation can be presented in the following form

$$\begin{aligned} R_{12}(\mu-\lambda)K_1^+(\lambda)R_{21}(-\lambda-\mu-2\eta)K_2^+(\mu) = \\ K_2^+(\mu)R_{12}(-\lambda-\mu-2\eta)K_1^+(\lambda)R_{21}(\mu-\lambda). \end{aligned} \tag{4}$$

One can then verify that the mapping

$$K^+(\lambda) = K^-(-\lambda-\eta) \tag{5}$$

is a bijection between solutions of the reflection equation and the dual reflection equation. After substitution of (5) into the dual reflection equation (4) one gets the reflection equation (3) with shifted arguments.

The general, spectral parameter dependent, solutions of the reflection equation (3) and the dual reflection equation (4) can be written as follows [26, 27, 28]

$$K^-(\lambda) = \begin{pmatrix} \kappa^- \sinh(\xi^- + \lambda) & \psi^- \sinh(2\lambda) \\ \phi^- \sinh(2\lambda) & \kappa^- \sinh(\xi^- - \lambda) \end{pmatrix}, \tag{6}$$

$$K^+(\lambda) = \begin{pmatrix} \kappa^+ \sinh(\xi^+ - \lambda - \eta) & -\psi^+ \sinh\left(2(\lambda+\eta)\right) \\ -\phi^+ \sinh\left(2(\lambda+\eta)\right) & \kappa^+ \sinh(\xi^+ + \lambda + \eta) \end{pmatrix}. \tag{7}$$

Due to the fact that the reflection matrices $K^\mp(\lambda)$ are defined up to multiplicative constants the values of parameters $\kappa^\mp$ are not essential, as long as they are different from zero. Although the R-matrix (1) has the $U(1)$ symmetry the reflection matrices $K^\mp(\lambda)$ (6) and (7) cannot be brought to the upper triangular form by the symmetry transformations like in the case of the XXX Heisenberg spin chain [14].

We will not discuss the Lax operators and the corresponding monodromies, relevant for the study of the spin $\frac{1}{2}$ XXZ Heisenberg spin chain [16]. Our aim is to obtain the R and K matrices corresponding the spin 1 XXZ Heisenberg chain. As our first step we use the fusion procedure [29, 21] in the space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ with the aim of deriving the spin 1

XXZ R-matrix. To this end we observe that the R-matrix (1) at $\lambda = -\eta$ is proportional to the projector onto the antisymmetric subspace of the space $\mathbb{C}^2 \otimes \mathbb{C}^2$,

$$\mathrm{P}^- = \frac{-1}{2\sinh(\eta)} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

For our purpose, the relevant projector is the complementary one, which projects onto the symmetric subspace of the space $\mathbb{C}^2 \otimes \mathbb{C}^2$,

$$\mathrm{P}^+ = \mathbb{1} - \mathrm{P}^-. \tag{9}$$

We obtain the spin 1 representation in the tensor product of the last two spaces,

$$R_{1(34)}(\lambda) = P^+_{34} R_{14}(\lambda - \eta) R_{13}(\lambda) P^+_{34}. \tag{10}$$

It straightforward the check that the corresponding Yang-Baxter equation is satisfied

$$R_{12}(\lambda - \mu) R_{1(34)}(\lambda) R_{2(34)}(\mu) = R_{2(34)}(\mu) R_{1(34)}(\lambda) R_{12}(\lambda - \mu). \tag{11}$$

Analogously, we have

$$R_{(12)3}(\lambda) = P^+_{12} R_{13}(\lambda) R_{23}(\lambda + \eta) P^+_{12}. \tag{12}$$

Finally, we obtain the relevant R-matrix

$$R_{(12)(34)}(\lambda) = P^+_{34} R_{(12)4}(\lambda - \eta) R_{(12)3}(\lambda) P^+_{34}. \tag{13}$$

In this case, the fusion procedure for the K-matrix [30] yields

$$K^-_{(12)}(\lambda) = P^+_{12} K^-_1(\lambda) R_{12}(2\lambda + \eta) K^-_2(\lambda + \eta) P^+_{12}. \tag{14}$$

A direct calculation shows that the corresponding reflection equation is satisfied

$$\begin{aligned} &R_{(12)(34)}(\lambda - \mu) K^-_{(12)}(\lambda) R_{(34)(12)}(\lambda + \mu) K^-_{(34)}(\mu) = \\ &= K^-_{(34)}(\mu) R_{(12)(34)}(\lambda + \mu) K^-_{(12)}(\lambda) R_{(34)(12)}(\lambda - \mu). \end{aligned} \tag{15}$$

Therefore we have achieved our main objective in obtaining the relevant R and K matrices. In the following section we will look at their explicit form and some important properties.

3. Spin 1 XXZ Heisenberg chain

In the appropriate bases the R-matrix (13) of the spin one XXZ Heisenberg chain can be represented as the following 9×9 matrix

$$R(\lambda,\eta) = \left(\begin{array}{ccc|ccc|ccc} a_1 & & & & & & & & \\ & a_2 & & b_1 & & & & & \\ & & a_3 & & b_2 & & b_3 & & \\ \hline & b_1 & & a_2 & & & & & \\ & & b_2 & & a_4 & & b_2 & & \\ & & & & & a_2 & & b_1 & \\ \hline & & b_3 & & b_2 & & a_3 & & \\ & & & & & b_1 & & a_2 & \\ & & & & & & & & a_1 \end{array}\right), \tag{16}$$

where the entries are given by

$$\begin{aligned} a_1 &= \sinh(\lambda+\eta)\sinh(\lambda+2\eta), \quad b_1 = \sinh(\lambda+\eta)\sinh(2\eta), \\ a_2 &= \sinh(\lambda)\sinh(\lambda+\eta), \quad b_2 = \sinh^2(\lambda)\sinh(2\eta)\cosh(\eta), \\ a_3 &= \sinh(\lambda)\sinh(\lambda-\eta), \quad b_3 = \sinh(\eta)\sinh(2\eta)\sinh(\lambda)\sinh(\lambda+\eta), \\ a_4 &= \sinh(\lambda)\sinh(\lambda+\eta)+\sinh(\eta)\sinh(2\eta). \end{aligned}$$

This R-matrix satisfies the Yang-Baxter equation in the space $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

$$R_{12}(\lambda-\mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda-\mu), \tag{17}$$

and has the $U(1)$ symmetry

$$[h_1 + h_2, R_{12}(\lambda,\eta)] = 0, \tag{18}$$

where $h = \mathrm{diag}(1,0,-1)$. The similarity transformation

$$\mathrm{Ad}\exp(\alpha\lambda(h_1 - h_2))R_{12}(\lambda,\eta), \tag{19}$$

with $\alpha = \frac{1}{2}$, yields the $O(3)$ invariant form of this R-matrix [31, 32, 33]. The R-matrix 16 has some important properties such as regularity, unitarity, PT-symmetry and crossing symmetry. The regularity condition at $\lambda = 0$ reads

$$R(0,\eta) = \sinh(\eta)\sinh(2\eta)\mathcal{P}, \tag{20}$$

where $\mathcal{P}$ is the permutation matrix of $\mathbb{C}^3 \otimes \mathbb{C}^3$. The unitarity relation is

$$R_{12}(\lambda)R_{12}(-\lambda) = \rho(\lambda)\mathbb{1}, \tag{21}$$

here ρ is the following function

$$\rho(\lambda) = \sinh(\lambda+\eta)\sinh(\lambda+2\eta)\sinh(\lambda-\eta)\sinh(\lambda-2\eta). \tag{22}$$

The so-called PT-symmetry states

$$R_{12}^{t}(\lambda) = R_{12}(\lambda). \tag{23}$$

Finally, the R-matrix (16) has the following crossing symmetry property:

$$R(\lambda) = (\mathcal{J} \otimes \mathbb{1})\, R^{t_2}(-\lambda - \eta) \left(\mathcal{J}^{-1} \otimes \mathbb{1}\right), \tag{24}$$

where t_2 denotes the transpose in the second space and the matrix $\mathcal{J}$ is given by

$$\mathcal{J} = \left(\begin{array}{rrr} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{array} \right). \tag{25}$$

In the basis in which the R-matrix (13) takes the form (16) the K-matrix (14) is given by

$$K^{-}(\lambda, \eta) = \sinh(2\lambda + \eta) \left(\begin{array}{ccc} \mathrm{k}_{11} & \mathrm{k}_{12} & \mathrm{k}_{13} \\ \mathrm{k}_{21} & \mathrm{k}_{22} & \mathrm{k}_{23} \\ \mathrm{k}_{31} & \mathrm{k}_{32} & \mathrm{k}_{33} \end{array} \right), \tag{26}$$

where

$$\begin{aligned}
\mathrm{k}_{11} &= \kappa^{-2} \sinh(\xi^- + \lambda + \tfrac{\eta}{2}) \sinh(\xi^- + \lambda - \tfrac{\eta}{2}) + \psi^- \phi^- \sinh(2\lambda - \eta) \sinh(\eta), \\
\mathrm{k}_{22} &= \kappa^{-2} \sinh(\xi^- - \lambda + \tfrac{\eta}{2}) \sinh(\xi^- + \lambda - \tfrac{\eta}{2}) + \psi^- \phi^- \sinh(2\lambda - \eta) \sinh(2\lambda + \eta), \\
\mathrm{k}_{33} &= \kappa^{-2} \sinh(\xi^- - \lambda + \tfrac{\eta}{2}) \sinh(\xi^- - \lambda - \tfrac{\eta}{2}) + \psi^- \phi^- \sinh(2\lambda - \eta) \sinh(\eta), \\
\mathrm{k}_{12} &= \kappa^- \psi^- \sqrt{2}\sqrt{\cosh(\eta)} \sinh(\xi^- + \lambda - \tfrac{\eta}{2}) \sinh(2\lambda), \\
\mathrm{k}_{23} &= \kappa^- \psi^- \sqrt{2}\sqrt{\cosh(\eta)} \sinh(\xi^- - \lambda + \tfrac{\eta}{2}) \sinh(2\lambda), \\
\mathrm{k}_{13} &= \psi^{-2} \sinh(2\lambda - \eta) \sinh(2\lambda), \\
\mathrm{k}_{21} &= \kappa^- \phi^- \sqrt{2}\sqrt{\cosh(\eta)} \sinh(\xi^- + \lambda - \tfrac{\eta}{2}) \sinh(2\lambda), \\
\mathrm{k}_{32} &= \kappa^- \phi^- \sqrt{2}\sqrt{\cosh(\eta)} \sinh(\xi^- - \lambda + \tfrac{\eta}{2}) \sinh(2\lambda), \\
\mathrm{k}_{31} &= \phi^{-2} \sinh(2\lambda - \eta) \sinh(2\lambda).
\end{aligned} \tag{27}$$

The R-matrix (16) and the above K-matrix satisfy the reflection equation

$$R_{12}(\lambda - \mu) K_1^-(\lambda) R_{21}(\lambda + \mu) K_2^-(\mu) = K_2^-(\mu) R_{12}(\lambda + \mu) K_1^-(\lambda) R_{21}(\lambda - \mu). \tag{28}$$

The study of the spin 1 XXZ Heisenberg chain would require the dual reflection equation and the relevant K^+ matrix. Although, now this would be straightforward we will not proceed in this direction since our main aim is to derive the corresponding Gaudin model through the so-called quasi-classical limit [14, 16].

4. Spin 1 XXZ Gaudin model

The classical r-matrices are essential tools in the study of the Gaudin models [5, 4, 6, 7, 8, 9, 10]. To this end we observe the quasi-classical property of the R-matrix (16)

$$\frac{1}{\sinh^2(\lambda)} R(\lambda,\eta) = \mathbb{1} + \eta \mathrm{r}(\lambda) + \mathcal{O}(\eta^2), \tag{29}$$

where the classical r-matrix is given by [4, 6]

$$\mathrm{r}(\lambda) = \frac{2}{\sinh(\lambda)} \left(\begin{array}{ccc|ccc|ccc} \cosh(\lambda) & & & & & & & & \\ & & & 1 & & & & & \\ & & -\cosh(\lambda) & & 1 & & & & \\ \hline & 1 & & & & & & & \\ & & 1 & & & & 1 & & \\ & & & & & & & 1 & \\ \hline & & & & 1 & & -\cosh(\lambda) & & \\ & & & & & 1 & & & \\ & & & & & & & & \cosh(\lambda) \end{array} \right) + \coth(\lambda)\mathbb{1}. \tag{30}$$

As it is very well known[4, 6], the above r-matrix has the unitarity property

$$\mathrm{r}_{21}(-\lambda) = -\mathrm{r}_{12}(\lambda), \tag{31}$$

and it satisfies the classical Yang-Baxter equation

$$[\mathrm{r}_{13}(\lambda), \mathrm{r}_{23}(\mu)] + [\mathrm{r}_{12}(\lambda-\mu), \mathrm{r}_{13}(\lambda) + \mathrm{r}_{23}(\mu)] = 0. \tag{32}$$

Our objective is the non-periodic Gaudin model and therefore we also need the classical K-matrix [14, 15, 16] which is obtained from the K-matrix (26) by setting $\eta = 0$,

$$\mathrm{K}(\lambda) \equiv K^{-}(\lambda, 0). \tag{33}$$

A direct consequence of the equation (28) is the classical reflection equation [34, 35, 15]:

$$\begin{aligned} &\mathrm{r}_{12}(\lambda-\mu)\mathrm{K}_1(\lambda)\mathrm{K}_2(\mu) + \mathrm{K}_1(\lambda)\mathrm{r}_{21}(\lambda+\mu)\mathrm{K}_2(\mu) = \\ &= \mathrm{K}_2(\mu)\mathrm{r}_{12}(\lambda+\mu)\mathrm{K}_1(\lambda) + \mathrm{K}_2(\mu)\mathrm{K}_1(\lambda)\mathrm{r}_{21}(\lambda-\mu). \end{aligned} \tag{34}$$

It can be shown [15, 12] that by defining the non-unitary classical r-matrix

$$\mathrm{r}_{12}^{K}(\lambda,\mu) = \mathrm{r}_{12}(\lambda-\mu) - \mathrm{K}_2(\mu)\mathrm{r}_{12}(\lambda+\mu)\mathrm{K}_2^{-1}(\mu), \tag{35}$$

the classical Yang-Baxter equation (32) and (34) combine into one equation, the so-called generalized classical Yang-Baxter equation

$$\left[\mathrm{r}_{32}^{\mathrm{K}}(\nu,\mu), \mathrm{r}_{13}^{\mathrm{K}}(\lambda,\nu)\right] + \left[\mathrm{r}_{12}^{\mathrm{K}}(\lambda,\mu), \mathrm{r}_{13}^{\mathrm{K}}(\lambda,\nu) + \mathrm{r}_{23}^{\mathrm{K}}(\mu,\nu)\right] = 0. \tag{36}$$

As the classical r-matrix defines the Sklyanin linear bracket, an essential tool in the study of the periodic Gaudin model [5, 6, 7, 8, 9, 10], the non-unitary r-matrix (34), through the relevant linear bracket, defines the so-called generalized Gaudin algebra essential in the study of the non-periodic model [12, 15, 17, 18, 19] .

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