Velocity memory effect for gravitational waves with torsion*

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Abstract

We study the geodesic motion of massive test particle in the presense of the torsional plane-fronted (pp) wave in the three-dimensional (3D) gravity. The idea of this investigation is to test the appearance of the memory effect for the torsional waves. Our analysis discovers that the *velocity memory effect* happens for all waves that go to zero as, retarded time, u goes to infinity at sufficient fast rate.

1. Introduction

Does some observable change occur when a gravitational wave passes through a system of test particles in Minkowski spacetime? The answer is affirmative and is know as the gravitational memory [1, 2]. This is the effect that happens when a gravitational wave passes through a system of test particles, in asymptotically flat spacetime, which are initially at rest. If, after the passage of a gravitational wave, permanent displacement of particles occur we call this displacement memory effect [1, 2, 3, 4] and if particles have non-zero relative velocity we call it velocity memory effect [5, 6, 7]. This is important effect because it represents a possible experimental set up for the detection of gravitational waves and investigation of their properties.

In this article we will analyze the geodesic motion of massive test particles in the background of the pp wave with torsion. The reason why we undertook this investigation is to see is there a memory effect for gravitational waves with torsion in the Poincaré gauge theory [8, 9, 10]. To investigate this we will use the solutions with propagating torsion [11]. Important thing to note is that gravitational pp waves in 3D are solutions which do not exist without torsion [11], meaning that in the absence of torsion metric becomes trivial. This offers us an interesting opportunity to study the effects of torsion at the level of geodesic motion.

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The paper is organized as follows. First, we review pp waves without torsion in three dimensions and show that this solutions do not exist in 3D general relativity. After that, we analyze the pp waves with torsion in 3D Poincaré gauge theory of gravity. Next, we derive the geodesic equations for the metric of the pp wave with torsion. Unfortunately, the geodesic equations cannot be solved analytically except in a very special case, which is not interesting from the aspect of memory effect because it is not asymptotically flat. Due to this technical problem we had to solve geodesic equations numerically and results are given as plots of velocity in a function of retarded time u.

Conventions we are using are the following. The Latin indices $(i, j, ...)$ refer to the local Lorentz coordinates and run over $(0, 1, 2)$. The spacetime indices are denoted with letters of Greek alphabet. The contraction of vector with a form we label with $\rfloor$. The e^i is triad 1-form and the dual basis h_i is defined by the following equation $h_i \rfloor e^j = \delta_i^j$. For the Hodge dual we use the standard symbol *, and the Hodge dual of triad is $^\star e^i = \frac{1}{2}\varepsilon^{ijk} e_j e_k$. The exterior product of forms is implicit in all formulas.

2. The pp waves without torsion in three-dimensions

To better understand the nature of the pp waves with torsion we start with the Riemannian pp waves. For more details see Ref. [11].

The metric of the pp waves in Brinkmann coordinates is

$$ds^2 = 2du(Sdu + dv) - dy^2 \,, \tag{1a}$$

we, also, introduce an auxiliary function H

$$S = \frac{1}{2}H(u, y) \,. \tag{1b}$$

From the metric it is easy to derive the form of the triad e^i so that $ds^2 = \eta_{ij} e^i \otimes e^j$ holds, where η_{ij} is the half-null flat metric

$$\eta_{ij} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$$

The triad is given by

$$e^0 := du \,, \qquad e^1 := Sdu + dv \,, \qquad e^2 := dy \,, \tag{2a}$$

The dual frame h_i define by the requirement $h_i \rfloor e^j = \delta_i^j$, where $\rfloor$ is a label for contraction, reads

$$h_0 = \partial_u - S\partial_v \,, \qquad h_1 = \partial_v \,, \qquad h_2 = \partial_y \,. \tag{2b}$$

Starting from the general formula for the Riemann connection

$$\omega^{ij} := -\frac{1}{2}\Big[h^i \rfloor de^j - h^j \rfloor de^i - (h^i \rfloor h^j \rfloor de^k)e_k\Big]\,,$$

we derive that the only non-zero component is

$$\omega^{12} = -\partial_y S\, e^0\,. \tag{3a}$$

From the above connection the Riemannian curvature is easily obtained

$$R^{ij} = 2e^0 k^{[i} Q^{j]}\,, \tag{4a}$$

where $k^i = (0,1,0)$ is a null vector and

$$Q^i = \partial_y^2 S e^2 \delta_2^i\,. \tag{4b}$$

The Ricci 1-form $Ric^i := h_m \rfloor Ric^{mi}$ is given by

$$Ric^i = e^0 k^i Q\,, \qquad Q = h_i \rfloor Q^i = \frac{1}{2}\partial_y^2 H, \tag{5a}$$

while the scalar curvature is zero

$$R = 0\,. \tag{5b}$$

Up to this point discussion was valid for any theory of gravity, now we want to look at what happens in general relativity in 3D. Action of general relativity $I = -a_0 \int d^3x R$ leads to the vacuum field equations

$$2a_0 G^n{}_i = 0\,, \tag{6}$$

where $G^n{}_i$ is the Einstein tensor. Einstein equations, after substitution of the metric, give

$$\partial_y^2 H = 0\,, \tag{7}$$

which only has a trivial solution

$$H = C(u) + yD(u)\,,$$

due to the fact that the curvature identically vanishes and this solution is diffeomorphic to Minkowski spacetime. This is to be expected because general relativity in three-dimensions does not have propagating local degrees of freedom and consequently should not have a wave solution.

3. The pp waves with torsion

The theory we will consider is a quadratic Poincaré theory of gravity which generalizes general relativity with 6 additional quadratic terms and as many free parameters $a_0, \ldots, a_2$ and $b_0, \ldots, b_2$. We will not write the action interested reader can find more details in Ref. [11].

The pp wave with torsion is a generalization of Riemannian pp wave which is obtained under assumption that the triad field (2) remains unchanged, while the connection takes the form

$$\omega^{ij} = \tilde{\omega}^{ij} + \frac{1}{2}\varepsilon^{ij}{}_m k^m k_n e^n G\,, \tag{8a}$$

$$G := S' + K\,. \tag{8b}$$

The function $K = K(u, y)$ is added to account for the effect of torsion, which is seen from the following expression for torsion

$$T^i := \nabla e^i = \frac{1}{2} K k^i k_m {}^\star e^m\,. \tag{9}$$

The curvature 2-form, Ricci 1- form and Ricci scalar are given by

$$\begin{aligned}
R^{ij} &= \varepsilon^{ijm} k_m k^{n\star} e_n\, G'\,,\\
Ric^i &= \frac{1}{2} k^i k_m e^m G'\,,\\
R &= 0\,.
\end{aligned} \tag{10}$$

The geometric configuration defined by the triad field (2) and the connection (8) represents a generalized gravitational plane-fronted wave of GR_Λ, or the *torsion wave* for short. The vector field $k = \partial_v$ is the Killing vector for both the metric and the torsion; moreover, it is a null and covariantly constant vector field. This allows us to consider the solution (12) as a generalized pp-wave.

The field equations [11] are given by

$$\begin{aligned}
&a_0 G' - a_1 K' = 0\,, \qquad \Lambda = 0\,,\\
&K'' + m^2 K = 0\,, \qquad m^2 = \frac{a_0(a_1 - a_0)}{b_4 a_1}\,,
\end{aligned} \tag{11}$$

with $G = S' + K$ and $S = H/2$. The solution of this equations is

$$\begin{aligned}
K &= A(u)\cos my + B(u)\sin my\,,\\
\frac{1}{2}H &= \frac{a_1 - a_0}{a_0 m}\left(A(u)\sin my - B(u)\cos my\right) + h_1(u) + h_2(u)y\,.
\end{aligned} \tag{12}$$

As we already said the h_1 and h_2 do not contribute to the radiation part of the curvature and can be discarded as trivial solution. Consequently, when the torsion is not present the metric becomes trivial. This is to be expected since general relativity in three-dimensions is a theory without propagating local degrees of freedom. Because the metric is crucially related to the torsion we can extract information about the torsion already on the level of the metric and geodesic motion.

4. Motion of massive test particle

In this section we investigate the geodesic motion of massive test particle in the presence of the massive pp wave with torsion described in the previous section. The geodesic motion of the test particle is obtained by solving a geodesic equation in which appear Christoffel (Riemannian) connection. So first we have to find the Christoffel connection for the metric of the masive pp wave with torsion.

4.1. Christoffel connection

The Christoffel connection is easily derived from the metric using the well known formula

$$\tilde{\Gamma}^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\alpha} \left(\partial_\nu g_{\alpha\rho} + \partial_\rho g_{\alpha\nu} - \partial_\alpha g_{\nu\rho} \right) , \tag{13}$$

and its non-zero components are given by

$$\begin{aligned} \tilde{\Gamma}^{v}{}_{uu} &= \frac{1}{2}\partial_u H \,, \\ \tilde{\Gamma}^{v}{}_{uy} &= \frac{1}{2}H' \,, \qquad \tilde{\Gamma}^{v}{}_{yu} = \frac{1}{2}\partial_y H \,, \\ \tilde{\Gamma}^{y}{}_{uu} &= \frac{1}{2}\partial_y H \,. \end{aligned} \tag{14}$$

Let us note that existence of a non-trivial metric of the pp wave is due to the presence of torsion. This allows us to see effects of torsion on the level of metric and, consequently, in geodesic motion of test particles [12].

4.2. Geodesic equations

The geodesic equation for u is

$$\frac{d^2 u}{d\lambda^2} = 0 \,. \tag{15}$$

Consequently, we take $u \equiv \lambda$.

The equation for y reads

$$\ddot{y} + \frac{1}{2}\partial_y H = 0 \,, \tag{16a}$$

after substitution of explicit form of the function H it becomes

$$\ddot{y} + \frac{a_1 - a_0}{a_0}(A(u)\cos my + B(u)\sin my) = 0\,. \tag{16b}$$

The equation for v is

$$\ddot{v} + \frac{1}{2}\partial_u H + \partial_y H \dot{y} = 0\,, \tag{17}$$

or explicitly

$$\begin{aligned} &\ddot{v} + \frac{a_1 - a_0}{m a_0}(A'(u)\sin my - B'(u)\cos my) \\ &+\frac{a_1 - a_0}{a_0}(A(u)\cos my + B(u)\sin my)\dot{y} = 0\,, \end{aligned} \tag{18}$$

4.3. Velocity memory effect

The velocity memory effect is present in the case when functions $A(u)$ and $B(u)$ vanish for large u. For numerical calculations it is better to introduce functions $\bar{A}(u) = \frac{a_1 - a_0}{a_0}A$ and $\bar{B}(u) = \frac{a_1 - a_0}{a_0}B(u)$ which we will use later in the text instead of $A(u)$ and $B(u)$. The velocity changes as one changes initial conditions, so this is a true observable effect. We do not show plots for different initial conditions because we wanted the presentation to be as short as possible.

4.4. Shockwave case

In the shock wave case when functions $\bar{A}(u) = 0$ and $\bar{B}(u)$ vanishes exponentially $\bar{B}(u) = e^{-(u-10)^2}$ numerical solutions of the geodesic equations gives the plots [12]for the particle velocities $\dot{y}$ and $\dot{v}$ shown in the Figure 1.

4.5. Slow fall off

In the case when $\bar{A}(u) = 0$ and $\bar{B}(u) = 1/u$ numerical solutions lead to the following plots [12] for the particle velocities $\dot{y}$ and $\dot{v}$ shown in the Figure 2.

5. Conclusion

We investigated a motion of massive test particles in asymptotically flat pp wave spacetime with torsion. The meaning of this is that test particle is initially well described by a particle in Minkowski spacetime and at some point a pp wave passes by and at time-like infinity a particle is again well described by its motion in Minkowski spacetime. Consequently, properties of particles motion at initial time and at infinity can be consistently compared.

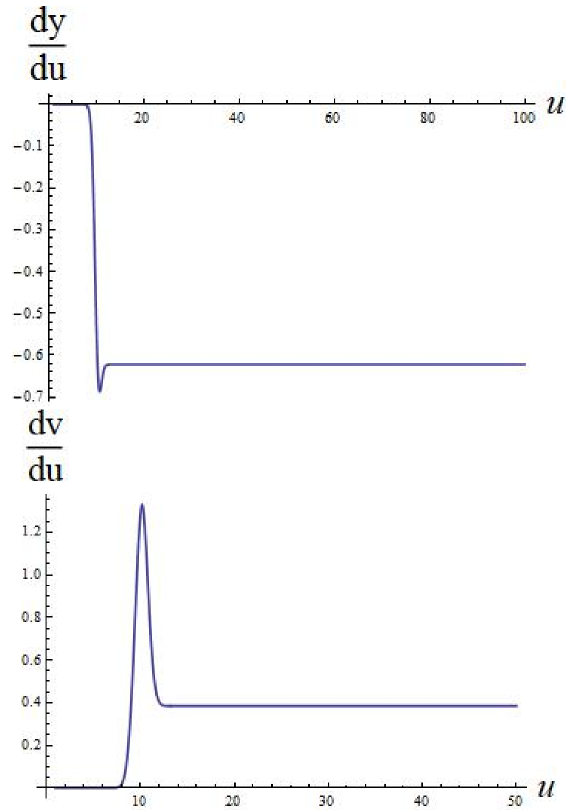

Figure 1: The plot for the particle velocity $\dot{y}$ and $\dot{v}$ in units $m = 1$, for $\bar{B} = -e^{-(u-10)^2}$

The conclusion is that velocity memory effect happens both for the exponentially fast fall-off of the gravitational wave as well as for the arbitrary polynomial fall-off. For the related work on memory effect for massive gravitons see Ref. [13]. This is the first time the memory effect for gravitational waves with torsion is analyzed. To authors knowledge, this is also the first example of the memory effect in three-dimensional gravity.

In the last few years there was a lot of effort on connecting asymptotic symmetries, soft theorems and memory effect [14]. This approach based on BMS symmetry offers a new perspective on the black hole microstates and information loss [15]. It is an open problem to connect the memory effect described in this article with asymptotic symmetry of the theory.

It is very interesting to generalize the analysis of this paper to the pp waves in four dimensions [16]. Preliminary results [17] show that most of the conclusions of this paper transfer to the four-dimensional case. This is important effect because it offers a possible experimental set up for the

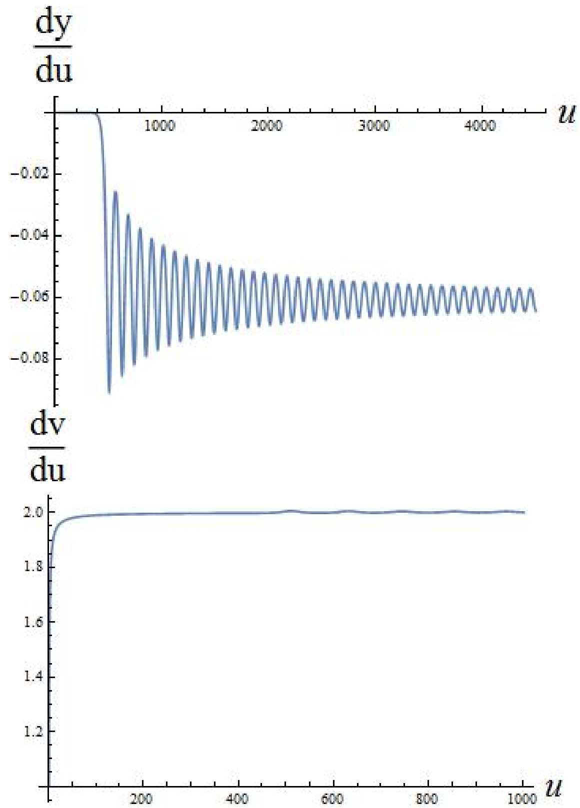

Figure 2: The plot for the particle velocity $\dot{y}$ and $\dot{v}$ in units $m = 1$, for $\bar{B} = -1/u$

detection of torsion.

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