Two type of contraction of conformal algebra and the gravity limit

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Abstract

Based on the results presented in previous work [1, 2] two types of Inönü-Wigner contraction [3] is presented. After going to gravitational limits [4] the same Lagrangian is obtained in both cases i.e. Holst action with cosmological constant. This simplification dose not requite additional constrains to obtain torsionless condition. One of the contracted algebra is extended by dilation generator, and because of this the final solution is invariant under conformal resacling.

1. Introduction

In the previous work [1, 2] we presented the constructions of MacDowell–Mansouri theory in BF formulation (A similar approch has been also presented e.g. in [4, 5]). We obtained the action which can be interpreted as an action of two Lorentzian co-frame one-forms. As a solution of equation of motion one could obtain the linear relation between two of them. Based of this one soldering form can be distinguished. Another consequence of field equation is that field associated with dilation generator is identically equal zero, thus it can be set to zero. The torsion does not vanish by field equation, therefore the additional constrain is required as discussed in [6]. After preceding the above steps you can stay with metric dependent Weyl action with Holst extension and the set of boundary terms [7, 8]. The final theory is invariant under conformal rescaling of the metric. The aim of this short paper is to check if the Inönü-Wigner contraction lead to the proper gravity action, as it has been done originally i.e the theory constructed based on contraction of anti de–Sitter symmetry $so(3,2)$ algebra to the Poincaré algebra. The result which can be expected is a gravity action, which can be invariant under rescaling. This is also the behavior of the final theory presented in [1, 9]. More over all terms containing scalar field disappear from theory because of contraction of algebra. Also the torsionless condition dose not have to be added as a separate constrain but it comes directly as solution of equation of motion. Just for simplicity the gravitation limit has been chosen as the simplest one i.e. It was assumed that two co-frame one-forms are equal.

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2. Construction of the conformal action

The construction of conformal invariant action starts from choosing $so(4,2)$ as a group of symmetry. It is well know that this Lie algebra is isomorphic to the conformal symmetry group $Conf(\mathbb{R}^{3,1})$ of the Minkowski space–time $\mathbb{R}^{3,1}$ by the relation [10, 11]:

$$J_{ij} = \mathcal{M}_{ij}, \qquad J_{4,5} = \mathcal{D}, \tag{1}$$

$$J_{4,j} = \frac{1}{2}(\mathcal{P}_j - \mathcal{K}_j) \qquad J_{5,j} = \frac{1}{2}(\mathcal{P}_j + \mathcal{K}_j)\ . \tag{2}$$

The $so(4,2)$ algebra is described by the following commutation relations:

$$[J_{IJ}, J_{KL}] = -i(\eta_{IK}J_{JL} + \eta_{JL}J_{IK} - \eta_{JK}J_{IL} - \eta_{IL}J_{JK})\ , \tag{3}$$

where $I, J = 0, \ldots, 5$ and η_{IJ} denotes a pseudo-Euclidean metric of the signature $(-,+,+,+,+,-)$. In matrix representation it can written as

$$J_{IJ} = \begin{pmatrix} \begin{pmatrix} \\ \mathcal{M}_{ij} \\ \\ \end{pmatrix} & \begin{pmatrix} \\ -\frac{1}{2}(\mathcal{P}_i - \mathcal{K}_i) \\ \\ \end{pmatrix} & \begin{pmatrix} \\ -\frac{1}{2}(P_i + K_i) \\ \\ \end{pmatrix} \\ \begin{pmatrix} \frac{1}{2}(\mathcal{P}_i - \mathcal{K}_i) \end{pmatrix} & 0 & -\mathcal{D} \\ \begin{pmatrix} \frac{1}{2}(\mathcal{P}_i + \mathcal{K}_i) \end{pmatrix} & \mathcal{D} & 0 \end{pmatrix}. \tag{4}$$

The choise of Anti de–Sitter algebra allows to use MacDowell–Mansouri mechanism [12] as it was done before by several authors. The mechanism is based on a gauge field with $so(4,2)$ gauge symmetry, which is intended to be broken down to Loretnz symmetry.

As a starting point one can construct a physical viable gauge connection field

$$A \equiv A^{IJ}J_{IJ} = \frac{1}{2}\omega^{ij}J_{ij} + \frac{1}{\ell}f_1^i J_{4i} + \frac{1}{\ell}f_2^i J_{5i} + \frac{1}{\ell}\phi J_{45}\,, \tag{5}$$

where $i, j = 0 \ldots 3$. Pulling-back this form to a four-dimensional space–time manifold M^4, one should notice that f_1^i, f_2^i can be interpreted as two Lorentzian co-frame one-forms. Additionally ϕ has interpretation as Lorentzian scalar one-form.

In the presented construction it is useful to rescale the generators by λ, κ i.e. const. complex number

$$J_{ij} = M_{ij}, \qquad J_{4i} = (\lambda - \kappa)R_{1i}, \qquad J_{5i} = (\lambda + \kappa)R_{2i}, \qquad J_{45} = \kappa D\,, \tag{6}$$

then the connection (5) becomes

$$A = A^{IJ}J_{IJ} = \frac{1}{2}\omega^{ij}M_{ij} + \frac{1}{\ell(\lambda - \kappa)}f_1^i R_{1i} + \frac{1}{\ell(\lambda + \kappa)}f_2^i R_{2i} + \frac{1}{\ell\kappa}\phi D\,. \tag{7}$$

The action is a MacDowell–Mansouri action [12, 13, 14, 15] on a 4-dimensional manifold M^4 in formulation of the BF theory [16, 17], and it takes the form

$$S = \int_{M^4} B^{IJ} \wedge F_{IJ} - \frac{\beta}{2} B^{IJ} \wedge B_{IJ} - \frac{1}{2} U^{MN} \epsilon_{IJKLMN} B^{IJ} \wedge B^{KL} , \quad (8)$$

where $I, J = 0 \dots 5$; $F \equiv F^{IJ} J_{IJ}$ is a curvature two-form for A (c.f. (5)) and the two-forms B_{IJ} are the components of a Lie algebra valued two-form $B \equiv B^{IJ} J_{IJ}$. To break symmetry one can assume the components of a scalar-valued matrix $U \equiv U^{MN} J_{MN}$ in the form:

$$U^{MN} = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \frac{\alpha}{2} \\ 0 & \cdots & -\frac{\alpha}{2} & 0 \end{pmatrix}$$

where α is a real, positive number.

The field equations for the auxiliary fields B^{ij}, B^{i4}, B^{i5} and B^{45} can be solved and put back to the action

$$S = \int_{M^4} \frac{1}{2} (Z_{ijkl})^{-1} F^{ij} \wedge F^{kl} + \frac{1}{\beta} (F^{i4} \wedge F_{i4} - F^{i5} \wedge F_{i5} - F^{45} \wedge F_{45}) \quad (9)$$

where $Z_{ij}{}^{kl} = (\beta \delta_{ij}{}^{kl} + \frac{\alpha}{2} \epsilon_{ij}{}^{kl})$. One can note that at acion (9) the Immirzi parameter [18, 19] has been introduced and it has sense as far as Immirzi parameter $\gamma^2 \neq -1$. At this approach parameters $\alpha, \beta, \ell, \gamma$ are related to physical (gravitational and cosmological) constants by the following relations [16, 17]

$$\frac{1}{\ell^2} = -\frac{\Lambda}{3} , \qquad \alpha = \frac{G\Lambda}{3(1+\gamma^2)} , \qquad \beta = \frac{\gamma G\Lambda}{3(1+\gamma^2)} . \quad (10)$$

Straightforward calculation can show that $\alpha \sim 10^{-120}$.

Using algebraic connection field (7) it is possible to calculate the curvature tensor explicitly in analogues to[9, 20]

$$F^{ij} M_{ij} = \Big(R^{ij} - \frac{1}{\ell^2 (\lambda - \kappa)^2} f_1^i \wedge f_1^j + \frac{1}{\ell^2 (\lambda + \kappa)^2} f_2^i \wedge f_2^j \Big) M_{ij} \quad (11)$$

$$F^{45} D = \Big(\frac{1}{\ell \kappa} d\phi - \frac{1}{\ell^2 (\lambda^2 - \kappa^2)} f_1^j \wedge f_{2j} \Big) D \quad (12)$$

$$F^{i4} R_{1i} = \Big(\frac{1}{\ell (\lambda - \kappa)} D^\omega f_1^i - \frac{1}{\ell^2 \kappa (\lambda + \kappa)} \phi \wedge f_2^i \Big) R_{1i} \quad (13)$$

$$F^{i4} R_{2i} = \Big(\frac{1}{\ell (\lambda + \kappa)} D^\omega f_2^i - \frac{1}{\ell^2 \kappa (\lambda - \kappa)} \phi \wedge f_1^i \Big) R_{2i} \quad (14)$$

where two-form R_{ij} is a space–time curvature of the corresponding Riemann-Cartan connection (see e.g. [13, 14, 15]). After substituting these curvature tensors into the action (9) one can obtain

$$\begin{aligned} S =& \int_{M^4} \frac{1}{4}\Big(\frac{\alpha}{\alpha^2+\beta^2}(\frac{\beta}{\alpha}\delta_{ijkl}-\epsilon_{ijkl})\Big)\big(R^{ij}\wedge R^{kl} - \frac{2}{\ell^4(\lambda^2-\kappa^2)^2} f_1^i\wedge f_1^j\wedge f_2^k\wedge f_2^l) \\ & - \frac{2}{\ell^2(\lambda-\kappa)^2}R^{ij}\wedge f_1^k\wedge f_1^l + \frac{2}{\ell^2(\lambda+\kappa)^2}R^{ij}\wedge f_2^k\wedge f_2^l \\ & + \frac{1}{\ell^4(\lambda-\kappa)^4} f_1^i\wedge f_1^j\wedge f_1^k\wedge f_1^l + \frac{1}{\ell^4(\lambda+\kappa)^4} f_2^i\wedge f_2^j\wedge f_2^k\wedge f_2^l \\ & - \frac{1}{\beta\ell^2}(\frac{1}{\kappa^2}d\phi\wedge d\phi - \frac{2}{\ell(\lambda^2-\kappa^2)\kappa}d\phi\wedge f_1^i\wedge f_{2i} - \frac{1}{\ell^2(\lambda^2-\kappa^2)^2} f_1^i\wedge f_{2i}\wedge f_1^j\wedge f_{2j}) \\ & + \frac{1}{\beta\ell^2(\lambda-\kappa)^2}\big(D^\omega f_1^i\wedge D^\omega f_{1i} - \frac{2}{l(\lambda^2-\kappa^2)\kappa}D^\omega f_1^i\wedge\phi\wedge f_{2i}\big) \\ & - \frac{1}{\beta\ell^2(\lambda+\kappa)^2}\big(D^\omega f_2^i\wedge D^\omega f_{2i} - \frac{2}{l(\lambda^2-\kappa^2)\kappa}D^\omega f_2^i\wedge\phi\wedge f_{1i}\big)\,. \end{aligned} \tag{15}$$

Similar form of the action has already appeared in [9, 20]. Some algebraic calculation can separate the boundary terms (i.a. Euler, Pontryagin and Nieh-Yan) form the action

$$\begin{aligned} S =& \int_{M^4} E_4 + P_4 + NY_4(f_1) - NY_4(f_2) \\ & + \frac{\epsilon_{ijkl}}{2G(\lambda+\kappa)^2}R^{ij}\wedge f_2^k\wedge f_2^l + \frac{\epsilon_{ijkl}}{4\ell^2G(\lambda+\kappa)^4} f_2^i\wedge f_2^j\wedge f_2^k\wedge f_2^l \\ & - \frac{\epsilon_{ijkl}}{2G(\lambda-\kappa)^2}R^{ij}\wedge f_1^k\wedge f_1^l + \frac{\epsilon_{ijkl}}{4\ell^2G(\lambda-\kappa)^4} f_1^i\wedge f_1^j\wedge f_1^k\wedge f_1^l \\ & - \frac{\epsilon_{ijkl}}{2G\ell^2(\lambda^2-\kappa^2)^2} f_1^i\wedge f_1^j\wedge f_2^k\wedge f_2^l \\ & + \frac{1}{2G\ell^2(\lambda^2-\kappa^2)^2} f_1^i\wedge f_1^j\wedge f_{2i}\wedge f_{2j} \\ & + \frac{1}{\beta\ell^4(\lambda^2-\kappa^2)^2} f_1^i\wedge f_{2i}\wedge f_1^j\wedge f_{2j} \\ & + \frac{1}{G\gamma(\lambda+\kappa)^2}R^{ij}\wedge f_{2i}\wedge f_{2j} - \frac{1}{G\gamma(\lambda-\kappa)^2}R^{ij}\wedge f_{1i}\wedge f_{1j} \\ & + \frac{2}{\beta\ell^3(\lambda^2-\kappa^2)\kappa^2}D^\omega f_1^i\wedge f_{2i}\wedge\phi - \frac{2}{\beta\ell^3(\lambda^2-\kappa^2)\kappa^2}D^\omega f_2^i\wedge f_{1i}\wedge\phi \\ & - \frac{1}{\beta\ell^2\kappa^2}d\phi\wedge d\phi + \frac{2}{\beta\ell^3(\lambda^2-\kappa^2)\kappa^2}d\phi\wedge f_1^i\wedge f_{2i}\,. \end{aligned} \tag{16}$$

At that point it is more convenient to use a conformal groups isomorphism and to change the basis in the algebra by the following transformation:

$$\frac{1}{(\lambda-\kappa)}f_1^i = \frac{1}{\lambda}e^i - \frac{1}{\kappa}f^i \tag{17}$$

$$\frac{1}{(\lambda+\kappa)}f_2^i = \frac{1}{\lambda}e^i + \frac{1}{\kappa}f^i\ . \tag{18}$$

One has to note that in conformal algebra some rescaling is made. It will turn out later that at gravitational limit this rescaling makes sens.

This choice of rescaling can be equivalent with taking rescaling of conformal algebra generators:

$$D \to \kappa D, \qquad P \to \lambda P, \qquad K \to \kappa K \tag{19}$$

A Lagrangian for the action (16) reads now as

$$\begin{aligned} S = \int_{M^4} & E_4 + P_4 + NY_4(e,f) \\ & + \frac{2\epsilon_{ijkl}}{G} R^{ij} \wedge e^k \wedge f^l + \frac{4\epsilon_{ijkl}}{\ell^2 G} e^i \wedge f^j \wedge e^k \wedge f^l \\ & + \frac{2}{G\ell^2} e^i \wedge e^j \wedge f_i \wedge f_j - \frac{4}{\beta\ell^4} e^i \wedge e^j \wedge f_i \wedge f_j \\ & + \frac{4}{G\gamma} R^{ij} \wedge e_i \wedge f_j \\ & + \frac{2}{\beta\ell^3\kappa} D^\omega(e^i \wedge f_i) \wedge \phi \\ & - \frac{1}{\beta\ell^2\kappa^2} d\phi \wedge d\phi + \frac{4}{\beta\ell^3\kappa^2} d\phi \wedge e^i \wedge f_i\ . \end{aligned} \tag{20}$$

one can notices that the equation (20) has been already obtained within a bigravity approach, as presented in [21]. Since the co–frame e^i is associated with the Poincaré translation generators (soldering form), the solution of equation of motion shows the linear realtion between fields e^i and f^i [9] i.e.

$$f^i_\mu = -6R^i_\mu + Re^i_\mu = \mathcal{F}\big(e(x)\big)e^i_\mu\ . \tag{21}$$

Equation (21) describe a special choice of gravitational limits i.e. $f^i = \mathcal{F}(x)e^i$, where $\mathcal{F}$ is some function of space–time. [1] It is also possible to make a simpler choice of function $\mathcal{F}$ i.e. $\mathcal{F}(x) = 1$ [4], and take limit $\kappa \to \infty$

[1] Some rescaling of field $f^i_\mu = -3R^i_\mu + \frac{1}{2}Re^i_\mu$ can give some impression that r.h.s. of this eq. is similar to r.h.s of Ricci flow equation.

the action (20) reads

$$S = \int_{M^4} E_4 + P_4 + NY_4 + \frac{2\epsilon_{ijkl}}{G} R^{ij} \wedge e^k \wedge e^l + \frac{4\epsilon_{ijkl}}{\ell^2 G} e^i \wedge e^j \wedge e^k \wedge e^l + \frac{4}{G\gamma} R^{ij} \wedge e_i \wedge e_j \,.$$

The equation of motion are standard Einstein equation and torsionless equation. The following gravity limit as well as limit of $\kappa \to \infty$ leads to two modification of conformal algebra for $\lambda = 1$

$$[\mathcal{D}, \mathcal{P}_i] = 0 \,, \tag{22}$$
$$[\mathcal{P}_i, \mathcal{P}_j] = -2i\eta_{ij}\mathcal{D} \,, \tag{23}$$
$$[\mathcal{M}_{ij}, \mathcal{P}_k] = -i(\eta_{ki}\mathcal{P}_j - \eta_{kj}\mathcal{P}_i) \,, \tag{24}$$
$$[\mathcal{M}_{ij}, \mathcal{M}_{kl}] = -i(\eta_{ik}\mathcal{M}_{jl} + \eta_{jl}\mathcal{M}_{ik} - \eta_{jk}\mathcal{M}_{il} - \eta_{il}\mathcal{M}_{jk}) \,, \tag{25}$$

and for $\lambda = \kappa$

$$[\mathcal{D}, \mathcal{P}_i] = 0 \,, \tag{26}$$
$$[\mathcal{P}_i, \mathcal{P}_j] = 0 \,, \tag{27}$$
$$[\mathcal{M}_{ij}, \mathcal{P}_k] = -i(\eta_{ki}\mathcal{P}_j - \eta_{kj}\mathcal{P}_i) \,, \tag{28}$$
$$[\mathcal{M}_{ij}, \mathcal{M}_{kl}] = -i(\eta_{ik}\mathcal{M}_{jl} + \eta_{jl}\mathcal{M}_{ik} - \eta_{jk}\mathcal{M}_{il} - \eta_{il}\mathcal{M}_{jk}) \,. \tag{29}$$

Algebra (26-29) is a Poincaré algebra as algebra (22-25) is Poincaré algebra extended by generator $\mathcal{D}$. Some one can note that Lagrangian dose not depend on λ any more, therefore it both cases Lagrangian is invariant under symmetries induced from both chose of algebras. Especially in case $\lambda = 1$ it is invariant under symmetry generated by $\mathcal{D}$.

3. Summary

The construction from [1] has been taken as starting point. But then the two types of very simple contractions has been presented on algebra level as well as action level. The obtained theory has a form of gravity with two co-frame feild without interaction term between them. Then gravitational limit has been proposed i.e. to make this two co-frame equal. The final theory is exactly the same action as obtained in [16, 17] i.e. Holst action with cosmological constant and boundary term [7, 8]. The main difference is that one of contraction contains trivial extension by dilation generator. Gauge symmetry base on it is conserved.

References

[1] A. Borowiec, M. Szczachor, to be published in AIPCP20-AR-TIM2019-00001

[2] M. Szczachor, Gravity and Supergravity as a BF theory of MacDowell-Mansouri type, PhD Thesis, Wrocław University, 27-10-2015.

[3] E. Inonu and E. P. Wigner, Proc. Nat. Acad. Sci. **39** (1953) 510.

[4] T. G. Zlosnik and H. F. Westman, arXiv:1601.00567 [gr-qc].

[5] A.B. Arbuzov, D.J. Cirilo-Lombardo, Int.J.Geom.Meth.Mod.Phys. 15 (2017) 1850005 (2017-10-05); arXiv:1708.00334 [hep-th]

[6] P. K. Townsend and P. van Nieuwenhuizen, Phys. Rev. D **19** (1979) 3166.

[7] S. Holst, Phys. Rev. D **53**, 5966 (1996). [gr-qc/9511026].

[8] D. J. Rezende and A. Perez, Phys. Rev. D **79**, 064026 (2009) [arXiv:0902.3416 [gr-qc]].

[9] M. Kaku, P. K. Townsend and P. van Nieuwenhuizen, Phys. Rev. D **17** (1978) 3179.

[10] G. Mack and A. Salam, Annals Phys. **53** (1969) 174.

[11] R. R. Metsaev, Mod. Phys. Lett. A **10** (1995) 1719.

[12] S. W. MacDowell and F. Mansouri, Phys. Rev. Lett. **38** (1977) 739 [Phys. Rev. Lett. **38** (1977) 1376].

[13] P. C. West, Phys. Lett.B7 1978, 6569.

[14] K.S. Stelle, P. C. West, J. Phys. A 12 (1979), no. 8, L205–L210.

[15] K. S. Stelle, P. C. West, Phys. Rev.D211,1980, 466.

[16] L. Freidel and A. Starodubtsev, hep-th/0501191.

[17] L. Smolin and A. Starodubtsev, hep-th/0311163.

[18] J. F. Barbero G., Phys. Rev. D **51** (1995) 5507 [gr-qc/9410014].

[19] G. Immirzi, Class. Quant. Grav. **14** (1997) L177 [gr-qc/9612030].

[20] M. Kaku, P. K. Townsend and P. van Nieuwenhuizen, Phys. Rev. Lett. **39** (1977) 1109.

[21] A. Borowiec, J. Kowalski-Glikman, M. Szczachor, Recent Developments in Theoretical and Experimental General Relativity, Astrophysics, and Relativistic Field Theories, Part B, Ed. by: K. Rosquist, R. T Jantzen, R. Ruffini, The Thirteenth Marcel Grossmann Meeting, pp. 1146-1148, World Scientific Publishing 2015; arXiv:1212.6661 [gr-qc]