Transformation properties of nonlinear evolution equations in 1+1 dimensions*

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Abstract

We study transformation properties of the general class of evolution equations $u_t = H(t, x, u, u_x, u_{xx})$, $H_{u_{xx}} \neq 0$ and construct a chain of its nested normalized subclasses. A special attention is paid to a class of variable coefficient equations of reaction-diffusion–convection type. The obtained equivalence groups can further be used in group analysis of such classes of nonlinear evolution equations including the construction of exact solutions and local conservation laws.

1. Introduction

A number of mathematical models in physics and biology are represented by (1+1)-dimensional second-order nonlinear evolution equations. Such models are used in such diverse fields as quantum field theory [1, pp. 294–341], physics of nanowire semiconductor devices [2], and population genetics [3].

Many nonlinear partial differential equations (PDEs) that are important for applications are parameterized by arbitrary elements (constants or functions) and constitute classes of PDEs. An important task is to study transformation properties of such classes, i.e. to describe explicitly nondegenerate point transformations that link fixed equations from the class (it is also possible to extend the consideration to contact transformations). If two PDEs are connected by such a transformation, then associated objects like exact solutions, local conservation laws, and various kinds of symmetries of these equations are also related by the respective transformation. Such connected equations are called equivalent or similar [4]. In particular, the equivalence method allows one to construct exact solutions for variable coefficient PDEs using known exact solutions for their constant coefficient

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counterparts, see, e.g., [5, 6, 7]. At the same time, nondegenerate point transformations appear to be a useful tool not only for finding exact solutions but also for exhaustive solving group classifications problems (see, e.g., [5, 6, 7, 8, 9, 10]), design of physical parameterization schemes [11], and study of integrability [12, 13, 14, 15].

The systematic investigation of transformation properties of classes of nonlinear PDEs was initiated in [16, 17]. Roughly speaking, an admissible transformation is a triple consisting of two fixed equations from a class and a transformation that links these equations. The set of admissible transformations considered with the standard operation of composition of transformations has a groupoid structure and therefore it is also called the equivalence groupoid [11]. Equivalence transformations, which are invaluable tools of group analysis of differential equations, generate a subset in a set of admissible transformations and always form a group. An equivalence transformation applied to any equation from the class always maps it to another equation from the same class, while an admissible transformation may exist only for a specific pair of equations from the class under consideration.

The usual equivalence group consists of the nondegenerate point transformations of the independent and dependent variables and of the arbitrary elements of the class, where transformations for independent and dependent variables are projectable on the space of these variables [4]. If the transformations for independent and/or dependent variables involve arbitrary elements, then the corresponding equivalence group is called generalized one. If new arbitrary elements appear to depend on old ones in a nonlocal way (e.g., new arbitrary elements are expressed via integrals of old ones), then the corresponding equivalence group is called extended one. Simultaneously weakening the conditions of locality and projectability, leads to the notion of extended generalized equivalence group.

If any admissible transformation in a given class is induced by a transformation from its usual equivalence group, then this class is called *normalized* in the usual sense. In analogous way, the notions of normalization of a class in generalized, extended and extended generalized senses are formulated [8]. It is important that normalized classes of differential equation have simpler transformation properties than non-normalized ones. Moreover, once it is proved that certain class is normalized, finding the equivalence groupoids of its subclasses becomes essentially simpler since they are always subgroupoids of the equivalence groupoid of the initial class.

We aim to study transformation properties of the general class of nonlinear second-order evolution equations of the form

$$u_t = H(t, x, u, u_x, u_{xx}), \quad H_{u_{xx}} \neq 0, \tag{1}$$

to construct a chain of its nested normalized subclasses and to find their equivalence groupoids. Then the obtained results is used to find equivalence groups for non-normalized classes of reaction–diffusion equations that are important for applications.

2. Admissible transformations of evolution equations

Any nondegenerate point transformation $\mathcal{T}$ relating two fixed equations $u_t=H$ and $\tilde u_{\tilde t}=\tilde H$ from the class (1) has the form $\tilde t=T(t)$, $\tilde x=X(t,x,u)$, $\tilde u=U(t,x,u)$ with $T_t(X_xU_u-X_uU_x)\neq 0$ [16, 18]. The partial derivatives are transformed as follows:

$$\tilde u_{\tilde t}=\frac{D_tU D_xX-D_xU D_tX}{T_tD_xX},\quad \tilde u_{\tilde x}=\frac{D_xU}{D_xX},\quad \tilde u_{\tilde x\tilde x}=\frac{1}{D_xX}D_x\left(\frac{D_xU}{D_xX}\right),$$

where $D_t=\partial_t+u_t\partial_u+u_{tt}\partial_{u_t}+u_{tx}\partial_{u_x}+\dots$ and $D_x=\partial_x+u_x\partial_u+u_{tx}\partial_{u_t}+u_{xx}\partial_{u_x}+\dots$ are operators of the total differentiation with respect to t and x. Moreover, it was proved in [19] that class (1) is normalized. For further consideration we use the following statement.

Theorem 1 ([19])**.** *Class* (1) *is normalized in the usual sense. Its equivalence group is formed by the transformations*

$$\tilde t=T(t),\quad \tilde x=X(t,x,u),\quad \tilde u=U(t,x,u),\tag{2}$$

$$\tilde H=\frac{X_xU_u-X_uU_x}{T_tD_xX}H+\frac{U_tD_xX-X_tD_xU}{T_tD_xX},\tag{3}$$

where $T_t(X_xU_u-X_uU_x)\neq 0$,

Formula (3) implies that the subclass of class (1) singled out by the condition $H_{u_{xx}u_{xx}}=0$ has the same equivalence transformation components for variables. The following statement is true.

Theorem 2. *The class of quasilinear second-order evolution equations,*

$$u_t=G(t,x,u,u_x)u_{xx}+F(t,x,u,u_x),\quad G\neq 0,\tag{4}$$

is normalized in the usual sense. Its equivalence group is formed by the transformation components for variables (2) *and the transformations for arbitrary elements*

$$\tilde G=\frac{(D_xX)^2}{T_t}G,\quad \tilde F=\frac{X_xU_u-X_uU_x}{T_tD_xX}F+\frac{U_tD_xX-X_tD_xU}{T_tD_xX}+$$
$$\frac{(X_{xx}+2X_{xu}u_x+X_{uu}u_x^2)D_xU-(U_{xx}+2U_{xu}u_x+U_{uu}u_x^2)D_xX}{T_tD_xX}G.$$

The transformation component for G implies that, if G does not depend on u_x, then $X_u=0$. We formulate more generally Lemma 1 from [20].

Theorem 3. *The class*

$$u_t=G(t,x,u)u_{xx}+F(t,x,u,u_x),\quad G\neq 0,\tag{5}$$

is normalized in the usual sense. Its equivalence group comprises the transformations

$$\tilde t = T(t), \quad \tilde x = X(t,x), \quad \tilde u = U(t,x,u), \quad \tilde G = \frac{X_x^2}{T_t}G, \tag{6}$$

$$\tilde F = \frac{U_u}{T_t}F + \frac{U_tX_x - X_tD_xU}{T_tX_x} + \frac{X_{xx}D_xU - (U_{xx} + 2U_{xu}u_x + U_{uu}u_x^2)X_x}{T_tX_x}G,$$

where $T_tX_xU_u \neq 0$.

Particular important subclass of class (5) is one, where the function F is polynomial in u_x and especially when it is quadratic or linear in u_x. We formulate separate statements for each of these cases.

Theorem 4. *The class*

$$u_t = G(t,x,u)u_{xx} + \sum_{k=0}^{n} F^k(t,x,u)u_x^k, \ n \geq 2, \quad G \neq 0, \tag{7}$$

is normalized in the usual sense. Its equivalence group consists of the transformations (6) *and the transformation components for the arbitrary elements* F^k, $k = 0,\dots,n$, *are found as solutions of the algebraic system resulting from the splitting of the following equation with respect to different powers of* u_x

$$\sum_{k=0}^{n} \tilde F^k \left(\frac{U_u}{X_x}u_x + \frac{U_x}{X_x}\right)^k = \frac{1}{T_tX_x}\big[X_xU_u\sum_{k=0}^{n}F^ku_x^k + U_tX_x - X_tD_xU + \big(X_{xx}D_xU - (U_{xx} + 2U_{xu}u_x + U_{uu}u_x^2)X_x\big)\,G\big].$$

Theorem 5. *The class*

$$u_t = G(t,x,u)u_{xx} + F^2(t,x,u)u_x^2 + F^1(t,x,u)u_x + F^0(t,x,u), \quad G \neq 0, \tag{8}$$

is normalized in the usual sense. Its equivalence group is formed by the transformations

$$\tilde t = T(t), \quad \tilde x = X(t,x), \quad \tilde u = U(t,x,u),$$

$$\tilde G = \frac{X_x^2}{T_t}G, \quad \tilde F^2 = \frac{X_x^2}{T_tU_u^2}\left(U_uF^2 - U_{uu}G\right), \quad \tilde F^1 = \frac{1}{T_tU_u}\Big[X_xU_uF^1 + 2\frac{X_xU_x}{U_u}(U_{uu}G - U_uF^2) - X_tU_u + (X_{xx}U_u - 2U_{xu}X_x)G\Big],$$

$$\tilde F^0 = \frac{1}{T_t}\left[\frac{U_x^2}{U_u}F^2 - U_xF^1 + U_uF^0 + U_t + \left(2\frac{U_x}{U_u}U_{xu} - U_{xx} - \frac{U_x^2}{U_u^2}U_{uu}\right)G\right],$$

where $T_tX_xU_u \neq 0$.

If we consider the subclass of class (8) singled out by the condition $F^2=0$, then its equivalence group is a proper subgroup of the equivalence group of class (8). The constraints for the transformations are derived by setting $\tilde F^2=0$ and $F^2=0$ in Theorem 5. This implies that $U_{uu}=0$. The following statement is true.

Theorem 6. *The class*

$$u_t=G(t,x,u)u_{xx}+F^1(t,x,u)u_x+F^0(t,x,u),\quad G\neq 0, \tag{9}$$

is normalized in the usual sense. Its equivalence group comprises the transformations

$$\tilde t=T(t),\quad \tilde x=X(t,x),\quad \tilde u=U^1(t,x)u+U^0(t,x),\quad T_tX_xU^1\neq 0, \tag{10}$$

$$\tilde G=\frac{X_x^2}{T_t}G,\quad \tilde F^1=\frac{1}{T_tU^1}\left(X_xU^1F^1-X_tU^1+(X_{xx}U^1-2U^1_xX_x)G\right),$$

$$\tilde F^0=\frac{1}{T_t}\Bigg[U^1F^0-(U^1_xu+U^0_x)F^1+U^1_tu+U^0_t+ \left(2\frac{U^1_x}{U^1}(U^1_xu+U^0_x)-U^1_{xx}u-U^0_{xx}\right)G\Bigg].$$

Consider one more subclass of class (8) for which the condition $U_{uu}=0$ holds for admissible transformations. This is the subclass singled out by the condition $F^2=G_u$,

$$u_t=(G(t,x,u)u_x)_x+K(t,x,u)u_x+P(t,x,u),\quad G\neq 0. \tag{11}$$

This class can be written in the form

$$u_t=Gu_{xx}+G_uu_x^2+(G_x+K)u_x+P,$$

where the connections between arbitrary elements of the latter class and class (8) are given by the formulas $F^2=G_u$, $F^1=G_x+K$, $F^0=P$. It contains derivatives of G and they naturally appear in transformation components for arbitrary elements of the equivalence group. This group can be considered as usual one if we extend the tuple of arbitrary elements by new elements G_u and G_x. Then the transformations of these variables take the form

$$\tilde G_{\tilde x}=\frac{X_x}{T_t}G_x+2\frac{X_{xx}}{T_t}G,\quad \tilde G_{\tilde u}=\frac{X_x^2}{T_tU^1}G_u.$$

It is easy to see that the group is really usual one, representing the above class in the form

$$u_t=Gu_{xx}+G^1u_x^2+(G^2+K)u_x+P$$

with additional arbitrary elements $G^1=G_u$, and $G^2=G_x$.

Theorem 7. *Reparameterized class* (11) *is normalized in the usual sense. Its equivalence group is formed by the transformations*

$$\tilde t = T(t),\quad \tilde x = X(t,x),\quad \tilde u = U^1(t,x)u + U^0(t,x),\quad T_tX_xU^1\neq 0,$$
$$\tilde G = \frac{X_x^2}{T_t}G,\quad \tilde K = \frac{X_x}{T_t}\left[K - \left(\frac{X_{xx}}{X_x}+2\frac{U^1_x}{U^1}\right)G - 2(U^1_xu+U^0_x)\frac{G_u}{U^1} - \frac{X_t}{X_x}\right],$$
$$\tilde P = \frac{1}{T_t}\left[U^1P + \frac{(U^1_xu+U^0_x)^2}{U^1}G_u - (U^1_xu+U^0_x)(G_x+K) + U^1_tu + U^0_t + \left(2\frac{U^1_x}{U^1}(U^1_xu+U^0_x) - U^1_{xx}u - U^0_{xx}\right)G\right].$$

The subclass of class (11) singled out by the condition $K=0$,

$$u_t = (G(t,x,u)u_x)_x + P(t,x,u),\quad G\neq 0, \tag{12}$$

is not normalized anymore in contrast to its covering classes considered above.

Constraints for its admissible transformations are derived setting $K=0$ and $\tilde K = 0$ in the transformations adduced in the previous theorem, which results in the equation

$$2\left(\frac{U^1_x}{U^1}u + \frac{U^0_x}{U^1}\right)G_u + \left(\frac{X_{xx}}{X_x}+2\frac{U^1_x}{U^1}\right)G + \frac{X_t}{X_x} = 0.$$

The further constraints on forms of X, U^1 and U^0 depend on values of the function G. If G does not satisfy the equation of the form $(au+b)G_u + cG + d = 0$, where a, b, c and d are functions of t and x, then the point transformations between equations from this class necessarily satisfy the conditions $X_t = X_{xx} = U^1_x = U^0_x = 0$, and such a subclass of class (12) will be normalized in the usual sense. The whole class (12) is not normalized. We adduce its equivalence group in the following statement.

Theorem 8. *The equivalence group of class* (12) *is comprised of the transformations*

$$\tilde t = T(t),\quad \tilde x = \delta_1x+\delta_2,\quad \tilde u = U^1(t)u + U^0(t),\quad T_tU^1\delta_1\neq 0,$$
$$\tilde G = \frac{\delta_1^2}{T_t}G,\quad \tilde P = \frac{1}{T_t}\left(U^1P + U^1_tu + U^0_t\right).$$

The description of the entire equivalence groupoid of class (12) needs additional study (see Conclusion).

Classes of evolution equations with variable coefficients of u_t often appear in applications. That is why we additionally consider the generalization of equations (11) of the form

$$S(t,x)u_t = (G(t,x,u)u_x)_x + K(t,x,u)u_x + P(t,x,u),\quad SG\neq 0. \tag{13}$$

In particular, the classes of variable-coefficient diffusion–reaction equations $f(x)u_t = (g(x)A(u)u_x)_x + h(x)B(u)$ and diffusion–convection equations $f(x)u_t = (g(x)A(u)u_x)_x + h(x)B(u)u_x$ $(fgA \neq 0)$ are subclasses of this class.

In principle, the coefficient $S(t, x)$ can be gauged to one by the family of point transformation

$$\tilde{t} = t, \quad \tilde{x} = \int_{x_0}^{x} S(t, y)\,\mathrm{d}y, \quad \tilde{u} = u.$$

Nevertheless, we will consider class (13) separately since its transformational properties become more complicated in comparison with those of class (11). The following statement is true.

Theorem 9. *Any point transformation between two fixed equations from class* (13) *has the form* (10)*. Then the respective values of the arbitrary elements are related via the formulas*

$$\frac{\tilde{K} + \tilde{G}_{\tilde{x}}}{\tilde{S}} = \frac{X_x}{T_t S}\left[K + G_x + \left(\frac{X_{xx}}{X_x} - 2\frac{U^1_x}{U^1}\right)G - 2(U^1_x u + U^0_x)\frac{G_u}{U^1} - \frac{X_t}{X_x}S\right],$$

$$\frac{\tilde{G}}{\tilde{S}} = \frac{X_x^2}{T_t}\frac{G}{S}, \quad \frac{\tilde{P}}{\tilde{S}} = \frac{1}{T_t S}\left[U^1 P + \frac{(U^1_x u + U^0_x)^2}{U^1}G_u + (U^1_t u + U^0_t)S + \left(2\frac{U^1_x}{U^1}(U^1_x u + U^0_x) - U^1_{xx}u - U^0_{xx}\right)G - (U^1_x u + U^0_x)(K + G_x)\right].$$

It is obvious that transformation properties of class (13) become more complicated in comparison with those of class (11). Transformations are defined only for fractions of arbitrary elements. It is explained by the fact that this class admits peculiar gauge equivalence transformation (an equivalence transformation for which independent and dependent variables do not transform but only arbitrary elements). This is the transformation

$$\tilde{S} = Z(t, x, S), \quad \tilde{G} = \frac{G}{S}Z, \quad \tilde{K} = \frac{K}{S}Z - G\left(\frac{Z}{S}\right)_x, \quad \tilde{P} = \frac{P}{S}Z,$$

where Z is an arbitrary smooth function of its variables with $Z_S \neq 0$.

Theorem 10. *The equivalence group of class* (13) *comprises the transfor-*

mations

$$\tilde t = T(t), \quad \tilde x = X(t,x), \quad \tilde u = U^1(t,x)u + U^0(t,x), \quad T_tX_xU^1 \neq 0,$$
$$\tilde S = Z(t,x,S), \quad \tilde G = \frac{X_x^2}{T_t}\frac{G}{S}Z,$$
$$\tilde K = \frac{X_xZ}{T_tS}\left[K - \left(\frac{X_{xx}}{X_x} + 2\frac{U^1_x}{U^1}\right)G - 2\mathcal{U}\frac{G_u}{U^1} - \frac{X_t}{X_x}S\right] - \frac{X_x}{T_t}G\left(\frac{Z}{S}\right)_x,$$
$$\tilde P = \frac{Z}{T_tS}\Bigg[U^1P + \frac{\mathcal{U}^2}{U^1}G_u - \mathcal{U}(K+G_x) + (U^1_tu + U^0_t)S + \left(2\frac{U^1_x}{U^1}\mathcal{U} - U^1_{xx}u - U^0_{xx}\right)G\Bigg],$$

where $\mathcal{U} = U^1_xu + U^0_x$.

Class (13) can be regarded as normalized in the usual sense since we can present it in the form $Su_t = Gu_{xx} + G^1u_x^2 + G^2u_x + P$ with additional arbitrary elements $G^1 = G_u$, and $G^2 = K + G_x$.

We note that the subclass of class (13) singled out by the condition $K = 0$, i.e. the class

$$S(t,x)u_t = (G(t,x,u)u_x)_x + P(t,x,u), \quad SG \neq 0, \tag{14}$$

is not normalized. In contrast to the case of class (13) the coefficient S is essential for class (14). The gauge equivalence transformations are quite simple in this case, namely, each coefficient can be multiplied by a nonvanishing smooth function of t. The equivalence group of class (14) is wider than the equivalence group of its subclass with $S = 1$ (cf. Theorem 8). The following statement is true.

Theorem 11. *The equivalence group of class* (14) *consists of the transformations*

$$\tilde t = T(t), \quad \tilde x = X(x), \quad \tilde u = U^1(t)u + U^0(t), \quad T_tX_xU^1 \neq 0,$$
$$\tilde S = \psi(t)\frac{T_t}{X_x}S, \quad \tilde G = \psi(t)X_xG, \quad \tilde P = \frac{\psi(t)}{X_x}\left(U^1P + (U^1_tu + U^0_t)S\right),$$

where $\psi(t)$ *is a nonvanishing smooth function of its variable.*

3. Conclusions

The chain of nested subclasses of the general class of (1+1)-dimensional second-order nonlinear evolution equations is constructed. Transformation properties of the derived subclasses are investigated. For all the subclasses the widest possible equivalence groups are found. For those subclasses that are proved to be normalized the found equivalence groups give the exhaustive description of the respective equivalence groupoids of these classes. So, we firstly proved that classes (4), (5), (7)–(9), (11) and (13) are normalized and then looked for their equivalence groups, which lead to the complete description of the equivalence groupoids of these classes.

Finding the equivalence groupoids for the non-normalized classes is a difficult task since the determining equations for components of admissible transformations are nonlinear ones. That is why for such classes other techniques are needed like partition of the class into normalized subclasses and the method of furcate splitting [21]. In future work we plan to use these methods to find the entire equivalence groupoids of classes (12) and (14), that are of special interest for further applications. There are many models with application in physics and biology which are members of these classes, e.g. variable coefficient Fisher and Newell–Whitehead–Segel equations. The results of this paper will be used, in particular, to complete the group classification of the class of variable coefficient reaction–diffusion equations $f(x)u_t = (g(x)A(u)u_x)_x + h(x)B(u)$, $fgA_u \neq 0$.

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