Singularity resolution in fuzzy de Sitter cosmology

Maja Burić
University of Belgrade

M∩Φ Belgrade ∗ September 2019

Outline

One of the pending problems in theoretical physics is description of the structure of spacetime at small distances, i.e. quantization of gravity.

Since a straightforward quantization of gravitational field does not work, other ideas are developed. On the line of physics, the idea that there is an elementary substructure (e.g. strings), quantized by the usual methods, that yields quanta of spacetime. On the line of mathematics, that there is a mathematical structure, yet to be discovered, that generalizes the existing notions of geometry, or of symmetry, or etc, and provides einsteinean gravity in the large scale limit. Also there are many ideas in between, which combine quantum field theory with geometry.

Quantum spacetime should, or should have a potential, to solve two main problems of the current description of space, time and matter: singularities in GR, divergences in QFT.

In any case, we usually belive that spacetime at small scales has some kind of geometric structure, different from that of a manifold: perhaps discrete.

A discrete structure physicists are very familiar with is that of an algebra, e.g. the algebra of operators in quantum mechanics or a Lie algebra. In noncommutative geometry one assumes that spacetime is described by an algebra of operators, more precisely, a C^* algebra.

In order to describe matter fields and their equations of motion, one needs derivatives. Therefore (following Einstein) one would like to introduce noncommutative differential geometry and further, to identify it with quantum gravity. This structure might also be only effective or emergent.

There are several different approaches to noncommutative geometry: we work within the noncommutative frame formalism of Madore.

A paradigmatic example of a noncommutative space is the fuzzy sphere.

To construct it one uses its symmetries: we generalize that construction, with the motivation to obtain 4-dimensional noncommutative (fuzzy) spacetimes with spherical symmetry, and thus find examples of realistic configurations of the gravitational field (cosmology, black holes).

We define four dimensional fuzzy de Sitter space using the de Sitter group SO(1,4) and its unitary irreducible representations. The talk is based on the work with J. Madore, L. Nenadović and D. Latas, arXiv: 1508.06058, 1709.05158, 1903.08378.

Noncommutative space is an algebra $\mathcal{A}$ generated by a set of hermitian coordinates x^μ

$$[x^\mu, x^\nu] = i\bar{k} J^{\mu\nu}(x).$$

We either have an abstract position algebra or its concrete representation.

The structure of this space can be described by the spectra of coordinates. There are however other ways to understand/define a noncommutative space: its symmetries, a set of coherent states, and very importantly, its commutative limit.

Diffeomorphisms on a noncommutative space are functions on the algebra: obviously, changes of coordinates change their spectrum and we need to identify coordinates correctly. In the following we use synonimously terms 'fuzzy' and 'noncommutative', not presuming either discrete spectra or finite-dimensional representations.

Differential structure of $\mathcal{A}$ is given by the momentum algebra.
Momenta p_α define a set of vector fields e_α, the free falling frame

$$e_\alpha f = [p_\alpha, f].$$

Commutator satisfies the Leibniz rule.

On a commutative manifold the moving frame is given by $e_\alpha f = e_\alpha^\mu\,(\partial_\mu f)$,

$$p_\alpha = e_\alpha^\mu\,\partial_\mu, \quad e_\alpha^\mu = [p_\alpha, x^\mu]$$

so momenta are outside the coordinate algebra $\mathcal{A}$. The space of vector fields has dimension equal to dimension of spacetime.

In the noncommutative case we define, in analogy

$$e_\alpha^\mu = [p_\alpha, x^\mu], \quad g^{\mu\nu} = e_\alpha^\mu\, e_\beta^\nu\, \eta^{\alpha\beta},$$

with additional conditions to assure orthonormality of the moving frame.

Laplacian of a scalar function is defined naturally,

$$\Delta f = \eta^{\alpha\beta}[p_\alpha, [p_\beta, f]].$$

It is possible, and rather straightforward, to define differential-geometric quantities like connection, covariant derivative, curvature and torsion, by formulae analogous to those given in the Cartan's description of geometry. Therefore one can describe scalar, spinor and gauge fields as well as curved gravitational backgrounds.

Definition of the action includes the trace, and therefore, in order to obtain concrete or more detailed results, we need a representation of $\mathcal{A}$.

In short: coordinates with their relations define the position space and its algebraic properties, momenta and their relations define the differential-geometric structure of a fuzzy space.

One can extend the construction of the fuzzy sphere to all homogeneous spaces: here, we discuss 4-dimensional fuzzy de Sitter space.

In commutative case, de Sitter space can be defined as embedding

$$-v^2 + w^2 + x^2 + y^2 + z^2 = \frac{3}{\Lambda}$$

in flat 5-dimensional space

$$ds^2 = -dv^2 + dw^2 + dx^2 + dy^2 + dz^2.$$

It has maximal symmetry. We define fuzzy de Sitter space using the algebra of its symmetry group SO(1,4).

Algebra of the SO(1,4) group has 10 generators $M_{\alpha\beta}$

$$[M_{\alpha\beta}, M_{\gamma\delta}] = -i(\eta_{\alpha\gamma} M_{\beta\delta} - \eta_{\alpha\delta} M_{\beta\gamma} - \eta_{\beta\gamma} M_{\alpha\delta} + \eta_{\beta\delta} M_{\alpha\gamma})$$

$\alpha, \beta, ... = 0, 1, 2, 3, 4$; we use signature $\eta_{\alpha\beta} = \mathrm{diag}(+ - - - -)$.

It has two Casimir operators, quadratic and quartic

$$\mathcal{Q} = -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta},$$

$$\mathcal{W} = -W_\alpha W^\alpha, \quad W_\alpha = \frac{1}{8} \epsilon_{\alpha\beta\gamma\delta\eta} M^{\beta\gamma} M^{\delta\eta}.$$

The second Casimir relation, $\mathcal{W}$=const, is analogous to an embedding of the 4-dim commutative de Sitter space in 5 flat directions.

We introduce noncommutative coordinates as $x^\alpha = \ell\, W^\alpha$ and define fuzzy de Sitter space as a UIR of the de Sitter algebra. The quartic Casimir is related to the cosmological constant, $\eta_{\alpha\beta}\, x^\alpha x^\beta = 3/\Lambda\,.$

Coordinates x^α are quadratic in the group generators and in general do not close into a Lie or quadratic algebra under commutation:

$$[W^\alpha, W^\beta] = -\frac{i}{2}\,\epsilon^{\alpha\beta\gamma\delta\eta}\, W_\gamma M_{\delta\eta}\,.$$

However, one can show that in irreducible representations they generate the whole algebra via

$$i\,\mathcal{W} M^{\rho\sigma} = [W^\rho, W^\sigma] + \frac{1}{2}\,\epsilon^{\alpha\mu\rho\sigma\tau} W_\tau [W_\alpha, W_\mu].$$

This formula will later has the meaning of Fourier transformation.

There are two choices of momenta that give metric of the de Sitter space in the commutative limit. The simplest is

$$ip_0 = M_{04}\,, \qquad ip_i = M_{i4} + M_{0i}.$$

From the frame formalism follows that the line element is

$$ds^2 = d\tau^2 - e^{-2\tau}\,(dx^i)^2$$

where cosmic time τ is given by $\tau/\ell = \log(W_0 - W_4)$.

In the conformal group notation, $M_{i4} + M_{0i}$ are translations and M_{04} is dilatation. From

$$[iM_{04}, W_0 - W_4] = W_0 - W_4$$

we find that dilatation is canonically conjugate to the cosmic time, i.e, it can be identified with the hamiltonian.

Unitary irreducible representations of de Sitter group are known, found by Thomas, Newton and Dixmier. They are denoted by two quantum numbers (s, ρ) (sometimes by (s, $\nu = i\rho$) or (s, $q = \frac{1}{2} + i\rho$)) and fall into following categories:

- Principal continuous series: $\rho \geq 0$, $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

  $$\mathcal{Q} = -s(s+1) + \frac{9}{4} + \rho^2, \quad \mathcal{W} = s(s+1)(\frac{1}{4} + \rho^2)$$

- Complementary continuous series: $\nu \in R$, $|\nu| < \frac{3}{2}$, $s = 0, 1, 2 \ldots$

  $$\mathcal{Q} = -s(s+1) + \frac{9}{4} - \nu^2, \quad \mathcal{W} = s(s+1)(\frac{1}{4} - \nu^2)$$

- Discrete series: $s = \frac{1}{2}, 1, \frac{3}{2}, 2 \ldots, \quad q = s, s-1, \ldots 0 \text{ or } \frac{1}{2}$

  $$\mathcal{Q} = -s(s+1) - (q+1)(q-2), \quad \mathcal{W} = -s(s+1)q(q-1)$$

The principal continuous series has Hilbert space representations which are given in the Bargmann-Wigner space of the UIR's of the Poincaré group.

Hilbert space for $(\rho, s = \frac{1}{2})$ UIR is the space of Dirac bispinors $\psi(\vec{p})$ which satisfy the Dirac equation. In the Dirac representation,

$$\psi(\vec{p}) = \begin{pmatrix} \varphi(\vec{p}) \\ -\dfrac{\vec{p}\cdot\vec{\sigma}}{p_0 + m}\,\varphi(\vec{p}) \end{pmatrix} .$$

The scalar product is given by

$$(\psi, \psi') = \int \frac{d^3p}{2p_0}\,\psi^\dagger \gamma^0 \psi' = \int \frac{d^3p}{p_0}\,\frac{2m}{p_0 + m}\,\varphi^\dagger \varphi' .$$

Having an explicit representation, we can solve the eigenvalue problems of the coordinates and thus determine algebraic properties of the space. Because of the specific scalar product, hermiticity of coordinates is nontrivial.

Introducing the expressions for the generators, we find

$$W^0 = -\frac{1}{2m}\begin{pmatrix} (\rho - \frac{i}{2})p_i\sigma^i + i\,p_0^2\,\frac{\partial}{\partial p^i}\sigma^i & \epsilon^{ijk}p_0 p_i\,\frac{\partial}{\partial p^j}\,\sigma_k + \frac{3i}{2}p_0 \\ \epsilon^{ijk}p_0 p_i\,\frac{\partial}{\partial p^j}\,\sigma_k + \frac{3i}{2}p_0 & (\rho - \frac{i}{2})p_i\sigma^i + ip_0^2\,\frac{\partial}{\partial p^i}\,\sigma^i \end{pmatrix}$$

$$W^4 = -\frac{1}{2}\begin{pmatrix} ip_0\,\frac{\partial}{\partial p^i}\sigma^i & \epsilon^{ijk}p_i\,\frac{\partial}{\partial p^j}\,\sigma_k + \frac{3i}{2} \\ \epsilon^{ijk}p_i\,\frac{\partial}{\partial p^j}\,\sigma_k + \frac{3i}{2} & ip_0\,\frac{\partial}{\partial p^i}\sigma^i \end{pmatrix}$$

To obtain the spectrum of the embedding time W^0 we do not have to solve differential equations: from the matrix elements of $M_{\alpha\beta}$ one can easily find that the spectrum is discrete in all UIR's with eigenvalues $k(k+1) - k'(k'+1)$.

Because of symmetry, the spectra of spatial coordinates W^i and of W^4 are the same: as simpler, we solve the differential equation for W^4. W^4 commutes with the spatial rotations so we can take the ansatz of the form

$$\varphi(\vec{p}) = \frac{f(p)}{p}\,\varphi_{jm} + \frac{h(p)}{p}\,\chi_{jm}\,,$$

$$\varphi_{jm} = \begin{pmatrix} \sqrt{\frac{j+m}{2j}}\; Y^{m-1/2}_{j-1/2} \\ \sqrt{\frac{j-m}{2j}}\; Y^{m+1/2}_{j-1/2} \end{pmatrix}, \quad \chi_{jm} = \begin{pmatrix} \sqrt{\frac{j+1-m}{2(j+1)}}\; Y^{m-1/2}_{j+1/2} \\ -\sqrt{\frac{j+1+m}{2(j+1)}}\; Y^{m+1/2}_{j+1/2} \end{pmatrix},$$

Y_l^m are spherical harmonics in momentum space, $p = |\vec{p}|$, $j = \frac{1}{2}, \frac{3}{2}, \ldots$

The eigenvalue equation for the bispinor $\tilde{\psi}_{\sigma jm}$

$$W_4\,\tilde{\psi}_{\sigma jm} = \sigma\tilde{\psi}_{\sigma jm}$$

reduces to two coupled equations for spinors $\tilde{\varphi}_{\sigma jm}$.

Introducing $f = (x^2-1)^{1/4}\,\tilde{F}$, $h = (x^2-1)^{1/4}\,\tilde{H}$ and $x = \frac{p_0}{m} \in (1,\infty)$ we obtain a set of Legendre equations

$$(x^2-1)\,\frac{d^2\tilde{F}}{dx^2} + 2x\,\frac{d\tilde{F}}{dx} - \frac{j^2}{x^2-1}\,\tilde{F} = 2i\sigma(2i\sigma-1)\tilde{F}\,,$$

$$(x^2-1)\,\frac{d^2\tilde{H}}{dx^2} + 2x\,\frac{d\tilde{H}}{dx} - \frac{(j+1)^2}{x^2-1}\,\tilde{H} = 2i\sigma(2i\sigma-1)\tilde{H}$$

with relation between $\tilde{F}$ and $\tilde{H}$.

A regular solution to these equations exists for every real σ, and is expressed in terms of the associated Legendre functions:

$$f_{\sigma j} = A\,(x^2-1)^{\frac{1}{4}}\, P^{-j}_{-2i\sigma}(x), \quad h_{\sigma j} = A\,(2i\sigma - j - 1)(x^2-1)^{\frac{1}{4}}\, P^{-j-1}_{-2i\sigma}(x)$$

Eigenfunctions of W^4 are orthogonal and normalized to δ-function,

$$\big(\tilde\psi_{\sigma jm}, \tilde\psi_{\sigma' j' m'}\big) = 2A^*A' \frac{\Gamma(\frac{1}{2}-2i\sigma)\,\Gamma(\frac{1}{2}+2i\sigma')}{\Gamma(j+1-2i\sigma)\,\Gamma(j+1+2i\sigma')}\delta_{mm'}\,\delta_{jj'}\,\delta(\sigma-\sigma')\,.$$

Thus, the spectrum of W^4 and of all spatial coordinates W^i is continuous, the real line.

We can use the same ansatz to determine eigenvalues of the cosmic time, $e^{\tau/\ell} = W_0 - W_4$, i.e. to solve

$$(W_0 - W_4)\,\psi_{\lambda jm} = \lambda\psi_{\lambda jm}\,.$$

After some calculation, we find that in this case natural variable is $z = \sqrt{\frac{p_0-m}{p_0+m}} \in (0,1)$. Introducing

$$f = (\frac{2}{1-z^2})^{-i\rho}\, z^{j+\frac{1}{2}}\, F\,, \qquad h = (\frac{2}{1-z^2})^{-i\rho}\, z^{-j-\frac{1}{2}}\, H$$

we obtain a set of Bessel equations

$$z^2\,\frac{d^2F}{dz^2} + z\,\frac{dF}{dz} + \left(4\lambda^2 z^2 - j^2\right) F = 0$$
$$z^2\,\frac{d^2H}{dz^2} + z\,\frac{dH}{dz} + \left(4\lambda^2 z^2 - (j+1)^2\right) H = 0\,.$$

The regular solution to these equations is

$$f_{\lambda j} = C\,(\frac{2}{1-z^2})^{-i\rho}\sqrt{z}\,J_j(2\lambda z), \qquad h_{\lambda j} = iC\,(\frac{2}{1-z^2})^{-i\rho}\sqrt{z}\,J_{j+1}(2\lambda z).$$

Solutions exist for every real λ, but for λ and $-\lambda$ they are proportional, so apparently the spectrum is continuous with $\lambda \in (0,\infty)$.

However! each solution is normalizable, while they are not orthogonal. This formally comes from compactness of the interval $z \in (0,1)$ in which solutions are finite, and properties of the Bessel functions. It signals that not all $\lambda \in (0,\infty)$ belong to the spectrum, i.e. that operator $W_0 - W_4$, as defined, is not hermitian.

To check this statement, we go back to definition of the scalar product, and the action of $W_0 - W_4$ in the subspace of radial functions:

$$(\psi_{jm}, (W_0 - W_4)\,\psi'_{j'm'}) = -i\,\delta_{jj'}\delta_{mm'}\int_0^1 dz\left(F^*\frac{dH'}{dz} + H^*\frac{dF'}{dz}\right)$$

$$= i\,\delta_{jj'}\delta_{mm'}\int_0^1 dz\,\left(\frac{dF^*}{dz}H' + \frac{dH^*}{dz}F'\right) - i\delta_{jj'}\delta_{mm'}\left(F^*H' + H^*F'\right)\Big|_0^1\ .$$

Clearly, $W_0 - W_4 = (W_0 - W_4)^\dagger$ only up to nonvanishing boundary term: the domains $\mathcal{D}(W_0 - W_4)$ and $\mathcal{D}(W_0^\dagger - W_4^\dagger)$ are not equal, and the operator is not self-adjoint.

To find whether the self-adjoint extensions exist, one can determine the deficiency indices (n_+, n_-), i.e. the regular solutions to equations

$$(W_0 - W_4)\,\psi_{\pm jm} = \pm i\psi_{\pm jm}\,.$$

For each $\pm$ (and each j) one regular solution exists, and it is expressed in terms of the modified Bessel functions. For example,

$$F_+ = Cz^{-j} I_j(2z)\,, \quad H_+ = -Cz^{j+1} I_{j+1}(2z)\,.$$

The deficiency indices in each subspace of fixed j are (1,1), so one can define a self-adjoint extension of $W_0 - W_4$ by defining the appropriate boundary conditions. These render hermiticity, and reduce the initial Hilbert space to the subspace of physical states.

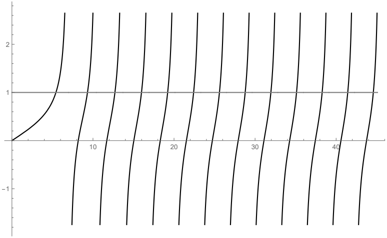
Using known mathematical procedure to find the boundary conditions, we obtain a one-parameter family (parametrized by c)

$$F(0) = H(0) = 0, \qquad H(1) = icF(1)\,.$$

Applied to the eigenfunctions of $W_0 - W_4$, this condition gives equation

$$\frac{J_{j+1}(2\lambda)}{J_j(2\lambda)} = c = \mathrm{const},$$

which has an infinite discrete set of solutions in λ.



We thus find that the spectrum of the cosmic time, when extended to a self-adjoint operator, is discrete. It is easy to check that the eigenfunctions which satisfy the additional boundary condition are orthogonal and span the physical Hilbert space.

Furthermore, when we calculate the expectation value of the radius of the universe, $r^2 = -\ell^2\, W_i W^i$, in the eigenstates of cosmic time, we find

$$\ell^2(\mathcal{W}+\lambda^2) \leq (\psi_{\lambda jm}, r^2\, \psi_{\lambda jm}) \leq \ell^2(\mathcal{W}+2\lambda^2)\,.$$

This means that the radius of the universe is bounded below by $\ell\sqrt{\mathcal{W}} = \ell\,\sqrt{\frac{3}{4}\,\left(\frac{1}{4}+\rho^2\right)}$, so there is no big bang singularity. For late times, radius grows exponentially, $r \sim \ell\lambda = \ell e^{t/\ell}$.

Necessity to introduce a self-adjoint extension gives spontaneous breaking of symmetry at low scales.

By careful analysis one can show that symmetry of the fuzzy metric which we use is in fact not $SO(1,4)$, but rather $U(1) \times SO(3)$. By a choice of one of the self-adjoint extensions, the $U(1)$-part of the symmetry is broken as eigenvalues are discrete. At intermediate values of time the lattice points become equidistant. At late times i.e. away from the Planck scale, the points are indistinguishable,

$$t_{n+1} - t_n \approx \ell \log \left(1 + \frac{1}{n}\right),$$

the time is (almost) continuous, and the original symmetry is effectively recovered.

Situations where self-adjointness is nontrivial appear also in quantum models of Liouville cosmology, representations of q-deformed Heisenberg algebra, of minimal-length Heisenberg algebra, etc.

Conclusions

- Fuzzy or noncommutative geometry alone gives desirable, interesting, new properties of spacetime in the quantum regime.
- For further effects in cosmology, one should study e.g. scalar fields on fuzzy gravitational backgrounds, which, in the formalism, is a well defined problem.