Special methods for solving nonlinear differential equations through polynomial expansions

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The paper proposes a generalized analytic approach which allows finding traveling wave solutions for some nonlinear PDEs. The solutions are expressed as polynomial expansions of the known solutions of an auxiliary equation. The proposed formalism integrates classical approaches as tanh method or G'/G *method, but it open the possibility of generating more complex solutions. A general class of second order PDEs is analyzed from the perspective of this formalism, and clear rules related to the balancing procedure are formulated. The KdV equation is used as a toy model to prove how the results obtained before through the* G'/G *approach can be recovered now in an unified and very natural way.*

Keywords: Traveling wave solutions, polynomial expansion, auxiliary equations, balancing procedure, KdV equation

- The paper is structured as follows: in the next section, basic facts on the polynomial expansion method are presented; in section three we shall focus on a general form of second order differential equation which includes important examples as Korteweg de Vries, Dodd-Boulogh-Mikhailov, nonlinear Schrodinger equation, nonlinear Klein-Gordon equation, Burger type equation, Fisher's equation, etc. We shall investigate the solutions of this general equation through the method we proposed. General balancing rules determining the order of the polynomials which have to be considered as solutions of the equation are established. In the fourth section the method is explicitly applied to the very simple model of the Korteweg de Vries equation. The solutions we are finding through our approach are compared with the solutions previously presented in literature. Final conclusions on the method will end the paper.

I. Introduction

- There is not a clear procedure to solve nonlinear PDEs, but, during the time, many analytic methods for finding exact solutions were formulated.
  - Cole and Hopf proposed a transformation method
  - Hirota used a bilinearization procedure
  - Ablowitz and Clarkson applied the inverse scattering formalism, etc.

- Important results have been pointed out by using various methods:
  - The symmetry method
  - Adomian decomposition method (efficient for equations which do not respond to other solving methods).

- An important class of nonlinear PDEs solutions is represented by the travelling waves. There are many direct methods for finding such solutions:

  the *tanh method*
  the *tanh-coth method*
  the *F-expansion method*
  the *exp-function method*
  the *elliptic function method*
  the G'/G*-method*

  We will focus on the last method supposing to look for solutions of an equation as an expansion in terms of the ratio G'/G. The method was generalized and improved.

- **MAIN QUESTION:** *Why* G'/G *and not* G'/G^2 *or any other expression can be considered? This is in fact the central aim of this work: to investigate how the* G'/G*-method can be further generalized and how all the mentioned methods for finding traveling wave solutions can be unified.*

II. The "auxiliary equation" method

- The algorithm of the **auxiliary equation method** is based on a very common approach supposing three main issues:
  (i) Reduction of the PDE to an ODE;
  (ii) Choice of an adequate auxiliary equation;
  (iii) Choice of a specific expression of the solutions for the investigated ODE in terms of those of the auxiliary equation.

- **Remarks:**
  - The first step, consisting in the reduction of the PDE to an ODE, is accomplished by introducing the wave variable. The auxiliary equation which will be attached to the resulting ODE is also an ODE but with well known solutions.
  - There are many choices of auxiliary equation which are proposed in literature, starting with Riccati equation [15], till more complex, higher order linear or nonlinear equations [14]. Depending on the choice of the auxiliary equation, the solutions of the studied equation can be, them too, simpler or more complex.
  - The novelty of our approach will be related to *(iii)* and it will consist in the fact that the solutions will be written in a general polynomial form, not related to a previously chosen ratio, as for example G'/G. There were previous attempts and results which pointed out solutions having different form as this ratio [Sheng], [Trav]. Our approach includes and extends these attempts.

Let us consider that the dependent variable $u(x,t)$, defined in a $2D$ space (x,t) satisfies the PDE:

$$F(u,u_t,u_x,u_{xx},u_{tt},\cdots)=0 \tag{1}$$

We consider the wave variable of the form:

$$\xi = x - Vt \tag{2}$$

Here V is a constant, identified as the wave velocity. With this transformation, the equation (1) becomes an ordinary differential equation:

$$\Delta(u,u',u'',\cdots)=0 \tag{3}$$

Here $u' = du(\xi)/d\xi$. We will look for a special class of analytical solutions of (3) which can be expressed as functions of the known solutions $G(\xi)$ of an auxiliary equation of the form:

$$A(G,G',G'',\cdots,G^{(k)})=0 \tag{4}$$

III. Choice of the solution

From the perspective of the auxiliary equation approach, the most general form of solution for (3) is:

$$u(\xi) = H(G, G', G'', \cdots G^{(k-1)}) \tag{5}$$

$H(G, G', G'', \cdots G^{(k-1)})$ designates a very general functional, containing $G(\xi)$ and its derivatives. Depending on the form of H, one can generate very complex solutions.

In almost all the cases, the auxiliary equations (4) have the simpler form $A(G, G', G'') = 0$, so the previous relation becomes:

$$u(\xi) = H(G, G') \tag{6}$$

The choice covering almost all the approaches currently used in literature is:

$$H(G, G') = h_0 G + h_1 G' + h_2 G'^2 + \ldots = \sum_{i=0}^{n} h_i G'^i . \tag{7}$$

If we will consider an expansion of the rhs of (6) we will get:

$$u(\xi) = \sum_{i=-m}^{m} P_i(G)(G')^i . \tag{8}$$

Here $P_i(G)$ are $2m+1$ functionals depending on $G(\xi)$, while m and n are constants to be determinate.

The generalized forms of the solution (6) or (8) we are proposing incorporate almost all representations of solutions which has been previously proposed by various authors. For example, the generalized and improved G'/G method [13], [14] corresponds to the choice:

$$h_0 \equiv d ; \quad h_1 \equiv 1$$

The representation used in [Zhou improved 2] is also included in (6), with: $H(G, G') = G' \sqrt{\sigma\left(1 + \frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)}$

IV. The auxiliary equation

The specific form of our generalized representations (6) and (8) also depends on the choice of the auxiliary equation (4).

Examples:

1) the *tanh method* uses an auxiliary a first order ODE equation of Riccati-type. In this case $H(G,G') = H(G) = \sum_{i=-m}^{m} a_i G^i$ The value of the parameter m depends on the model and it is established through a balancing procedure among the terms of higher order derivative, respectively of the higher nonlinearity.
2) Other examples of first order ODE used as auxiliary equation are:

$$F' = \frac{A}{F} + BF + CF^3$$
$$G' = c_2 G^2 + c_4 G^4 + c_6 G^6$$

3) If a *second order auxiliary equation* is considered, the second order derivative, G'' can be expressed in terms of G and G', and, again, the general solution (0.5) reduces to (0.5c) with $h_i = 0$ for $i \geq 2$. Examples of second order auxiliary equations are:

$$G'' + AG' + BG = 0$$
$$AGG'' - B(G')^2 - CGG' - EG^2 = 0$$

4) If we are dealing with equations of third order, the third order derivative can be in principle expressed in terms of the second and first orders, so it can be eliminated, and (5) stops at terms at maximum second order, G''.

V. The determining system for the polynomials

Coming back to our problem of finding solutions of a nonlinear PDE, let's summarize the method we are proposing:

- We use the wave variable and transform the PDE into an ODE:

$$u(x,t) \to u(\xi) \Rightarrow F(u, u_t, u_x, u_{xx}, u_{tt}, \cdots) = 0 \tag{9}$$

- We choose as auxiliary equation an ODE with known solutions, as for example:

$$G'' + AG' + BG = 0 \tag{10}$$

- We look for solutions of the form:

$$u(\xi) = \sum_{i=-m}^{m} P_i(G)(G')^i \tag{11}$$

- We determine the summation limit *m* by a balancing procedure following the powers of G'.
- From (9) and (11) we get a system of ODE for $P_i(G)$.
- For solving this system, we are looking to solutions of the form:

$$P_q(G) = k\frac{A_q(G)}{B_q(G)}, q = 1,\ldots,m \tag{12}$$

We will suppose that $A_q(G)$ **and** $B_q(G)$ **are polynomials**:

$$P_q(G) = \frac{\sum_{k=-N_{Aq}}^{N_{Aq}} a_{qk} G^k}{\sum_{k=-N_{Bq}}^{N_{Bq}} b_{qk} G^k}, \tag{13}$$

The finding of the unknown polynomials $P_q(G)$ supposes the finding of the coefficients a_{qk}, b_{qk}.

- We write down the solutions $u(\xi)$ in a final form.

Note: The expansions of the type (11) are in fact the most general possible form of solutions and they includes almost all the choices used in various approaches to the direct finding of exact solutions of nonlinear differential equations. It includes the ratio G'/G, which appears now in the most natural way.

VI. A generalized second order differential equation

Let us come back to the case when, after introducing the wave variable, an ODE is generated. Specifically, we will consider a large class of second order ODEs of the form:

$$A(u)u'' + B(u)u'^2 + C(u)u' + E(u) = 0 \tag{15}$$

Many very interesting and of practical interest equations, with applications in various fields, belong to (15). For example:

a) If $C(u) = 0$ we have: $$A(u)u'' + B(u)u'^2 + E(u) = 0. \tag{16}$$

To this category belongs Dodd-Boulogh-Mikhailov equation, describing fluid flows or QFT systems:

$$-Vuu'' + Vu'^2 + u^3 + 1 = 0$$

b) If $B(u) = 0, C(u) = 0$ the equation (15) becomes:

$$A(u)u'' + E(u) = 0. \tag{17}$$

Specific examples of equations belonging to this class are:

- Schrodinger equation with cubic nonlinearity: $u'' + u^3 - (\beta + \alpha^2)u = 0$

- Nonlinear Klein-Gordon equation: $k^2(\omega^2 - 1)u'' + \beta u^3 + \alpha u = 0$

- Benjamin - Bona - Mahony equation: $-bPu'' - au^2 - (1 + P)u + k = 0$

- Korteweg de Vries equation: $\delta u'' + \frac{1}{2}u^2 - Vu + k = 0$

c) When $B(u)=0,\ E(u)=0$ we get a Burger type equation:

$$A(u)u'' + C(u)u' = 0 \tag{18}$$

A common specific form is (ν =diffusion coefficient):

$$-Vu' + u\,u' = \nu u''$$

d) For $B(u)=0$ we get:

$$A(u)u'' + C(u)u' + E(u) = 0 \tag{19}$$

In particular, it can be the Chafee-Infante or Fisher's equations:

$$-u'' - Vu' + \lambda(u^3 - u) = 0$$

$$u'' - u' - u^2 + u = 0$$

e) If $C(u)=0,\ E(u)=0$, we obtain the Hunter-Saxon equation, which in particular has the form:

$$(u - V)u'' + \frac{1}{2}u'^2 = 0$$

This equation is important in the theoretical studies of liquid nematic crystals.

f) If $E(u)=0$ the equation (15) becomes:

$$A(u)u'' + B(u)u'^2 + C(u)u' = 0$$

In particular, it contains the Buckmaster equation, describing thin viscous fluid sheet flow:

$$4u^3u'' + 12u^2u'^2 + (3u^2 + V)u' = 0$$

VII. The example of the KdV Equation

To prove that our approach generalizes other classical methods for solving nonlinear PDEs, we will show how it allows recovering for the simpler case of KdV, the solutions that can be generated through G'/G method.

The Korteweg de Vries equation:

$$u_t + uu_x + \delta u_{xxx} = 0 \tag{20}$$

The wave variable is: $\xi = x - Vt$

By integrating once, we get the ODE:

$$\delta u''(\xi) + \frac{1}{2}u^2(\xi) - Vu(\xi) + k = 0 \tag{21}$$

Here δ, k, V are constants which will be used as parameters.

The balancing procedure between the terms $\delta u''(\xi)$ and $\frac{1}{2}u^2(\xi)$ leads to $m = 2$, so the solutions will have the expansion:

$$u(\xi) = \sum_{i=-2}^{2} P_i(G)(G')^i \tag{22}$$

We will suppose that the function $G(\xi)$ satisfy the auxiliary equation of the form:

$$G'' + AG' + BG = 0. \tag{23}$$

The determining system for the polynomials $\{ P_k(G),\ \ k=-2,-1,0,1,2 \}$

By vanishing the coefficients of various powers of G', we get the following system of equations ($\dot{P}_i \equiv \frac{dP_i}{dG}, \ddot{P}_i \equiv \frac{d^2 P_i}{dG^2}.$):

$$2\delta \ddot{P}_2(G)+P_2^2(G)=0 \tag{24}$$

$$\delta \ddot{P}_1(G)-5\delta \dot{P}_2(G)+P_1(G)P_2(G)=0 \tag{25}$$

$$\delta \ddot{P}_0(G)-3\delta A\dot{P}_1(G)-5\delta BG\dot{P}_2(G)+2\delta(2A^2-B)P_2(G)+\frac{1}{2}P_1^2(G)+P_0(G)P_2(G)-VP_2(G)=0 \tag{26}$$

$$-\delta A\dot{P}_0(G)-3\delta B\dot{P}_1(G)G+\delta(A^2-B)P_1(G)+6\delta ABGP_2(G)+P_0(G)P_1(G)-VP_1(G)=0 \tag{27}$$

$$-\delta BG\dot{P}_0(G)+\frac{1}{2}P_0^2(G)-VP_0(G)+\delta ABGP_1(G)+2\delta B^2G^2P_2(G)+k+\delta A\dot{P}_{-1}(G)+\delta \ddot{P}_{-2}(G)+P_1(G)P_{-1}(G)+P_2(G)P_{-2}(G)=0$$

$$P_{-2}(G)\left(\frac{1}{2}P_{-2}(G)+6\delta B^2G^2\right)=0 \tag{29}$$

$$10\delta ABGP_{-2}(G)+2\delta B^2G^2P_{-1}(G)+P_{-1}(G)P_{-2}(G)=0 \tag{30}$$

$$3\delta BG\dot{P}_{-2}(G)+3\delta ABGP_{-1}(G)+(4\delta A^2+2\delta B)P_{-2}(G)+P_0(G)P_{-2}(G)+\frac{1}{2}P_{-1}^2(G)-VP_{-2}(G)=0 \tag{31}$$

$$3\delta A\dot{P}_{-2}(G)+\delta A^2P_{-1}(G)+\delta B(P_{-1}(G)+G\dot{P}_{-1}(G))+P_0(G)P_{-1}(G)+P_1(G)P_{-2}(G)-VP_{-1}(G)=0 \tag{32}$$

Recovering the solutions given by the G'/G method

The system contains differential equations which belong to the class of equations (15), so they obey the general results related to the balancing procedure.
For the equation (24) we get $m=2$. The expression we are looking for P_2 will be:

$$P_2(G)=\sum_{i=-2}^{2} a_i G^i$$

Direct computations give:

$$P_2(G)=a_{-2}G^{-2}=-12\delta G^{-2} \tag{33}$$

The balancing procedure for the equation (25) leads to the solution:

$$P_1(G)=\sum_{j=-1}^{1} b_j G^j = b_{-1}G^{-1}=-12\delta A G^{-1} \tag{34}$$

Similarly, we get:

$$P_{-2}(G)=-12\delta B^2 G^2 \tag{35}$$

$$P_{-1}(G)=-12\delta ABG \tag{36}$$

$$P_0 = V-\delta A^2-8\delta B \tag{37}$$

The compatibility condition we have:

$$k=\frac{1}{2}P_0^2+\delta P_0 A^2+8\delta P_0 B-120\delta^2 A^2 B-96\delta^2 B^2 \tag{38}$$

The final solution given by our approach for the KdV equation (20) is:

$$u(\xi)=-12\delta B^2\left(\frac{G'}{G}\right)^{-2}-12\delta AB\left(\frac{G'}{G}\right)^{-1}+P_0-12\delta A\frac{G'}{G}-12\delta\left(\frac{G'}{G}\right)^{2} \tag{39}$$

More than the G'/G solutions

The solutions (33) – (38) are not the most general solutions of the determining system (24) – (32)!

It is quite simple to check that most general solutions exist.

If, for example, we choose:

$$P_2(G) = \frac{a_0}{b_0 + b_1 G + b_2 G^2}$$

Simple computations give:

$$P_1 = -\frac{24 b_2}{2 b_2 G + b_1}$$

$$P_0 = V - 2\delta\lambda^2$$

$$k = \frac{1}{2} P_0{}^2 + \delta P_0 \lambda^2$$

The general KdV solution becomes:

$$u(\xi) = V - 2\delta\lambda^2 - \frac{24 b_2 G'}{2 b_2 G + b_1} + \frac{a_0 (G')^2}{b_0 + b_1 G + b_2 G^2}$$

Specific solutions for KdV

Let us now analyze in a more specific way the general solution (33). As it was expected, it depends on the main parameters δ and V from KdV equation, as well as on A and B which appear in the auxiliary equation (23). Related to the last two, we know that three different situationa have to be considered:

(i) if $\Delta = A^2 - 4B > 0$ the auxiliary equation has a hyperbolic solution:

$$G(\xi) = e^{-(\lambda/2)\xi}\left(A_1 ch\frac{\sqrt{\Delta}}{2}\xi + A_2 sh\frac{\sqrt{\Delta}}{2}\xi \right).$$

$$u(\xi) = P_0 + 6\delta A\lambda - 6\delta A\sqrt{\Delta}\frac{A_1 sh\frac{\sqrt{\Delta}}{2}\xi + A_2 ch\frac{\sqrt{\Delta}}{2}\xi}{A_1 ch\frac{\sqrt{\Delta}}{2}\xi + A_2 sh\frac{\sqrt{\Delta}}{2}\xi} - 12\delta\left(-\frac{\lambda}{2} + \frac{\sqrt{\Delta}}{2}\frac{A_1 sh\frac{\sqrt{\Delta}}{2}\xi + A_2 ch\frac{\sqrt{\Delta}}{2}\xi}{A_1 ch\frac{\sqrt{\Delta}}{2}\xi + A_2 sh\frac{\sqrt{\Delta}}{2}\xi} \right)^2$$

(ii) if $\Delta = A^2 - 4B < 0$ the solution will be expressed through periodic functions:

$$G(\xi) = e^{-(\lambda/2)\xi}\left(A_1 \cos\frac{\sqrt{-\Delta}}{2}\xi + A_2 \sin\frac{\sqrt{-\Delta}}{2}\xi \right).$$

$$u(\xi) = P_0 + 6\delta A\lambda - 6\delta A\sqrt{-\Delta}\frac{-A_1 \sin\frac{\sqrt{-\Delta}}{2}\xi + A_2 \cos\frac{\sqrt{-\Delta}}{2}\xi}{A_1 \cos\frac{\sqrt{-\Delta}}{2}\xi + A_2 \sin\frac{\sqrt{-\Delta}}{2}\xi} - 12\delta\left(-\frac{\lambda}{2} + \frac{\sqrt{-\Delta}}{2}\frac{-A_1 \sin\frac{\sqrt{-\Delta}}{2}\xi + A_2 \cos\frac{\sqrt{-\Delta}}{2}\xi}{A_1 \cos\frac{\sqrt{-\Delta}}{2}\xi + A_2 \sin\frac{\sqrt{-\Delta}}{2}\xi} \right)^2$$

(iii) if $\Delta = A^2 - 4B = 0$ the solution will contain real exponentials:

$$G(\xi) = e^{-(\lambda/2)\xi}(A_1 + A_2\xi).$$

$$u(\xi) = P_0 + 6\delta A\lambda - 12\delta A\frac{A_2}{(A_1 + A_2\xi)} - 12\delta\left(-\frac{\lambda}{2} + \frac{A_2}{(A_1 + A_2\xi)} \right)^2$$

In the previous relations A_1 and A_2 denote arbitrary constants.

Conclusions

In this article, a generalized approach to the direct finding of the traveling wave solutions of the nonlinear differential equations has been considered. It consists in looking for solutions in the form of general polynomial expansions in terms of the solutions of an auxiliary equation. The approach includes all the classical methods which have been previously used and it creates premises for generating more complex solutions. The algorithm contains the standard steps: (i) transforming the PDE into an ODE through the wave variable; (ii) choosing an adequate auxiliary equation; (iii) representing the solution of the PDE as an expansion in terms of the solutions of the auxiliary equation. The novelty we brought is related to the third step. The procedure is purely analytic and it is very simple to be applied for any type of equation admitting traveling wave solutions.

We considered the example of a generalized class of second order PDEs containing many equations with important practical applications and an extensive study on the balancing procedure which limits the form of the possible solutions has been done. As specific example, the KdV equation is considered in details and, particularly, the solutions generated through the G'/G method have been recovered. Here are few remarks summarizing the results.

Remark 1: In general, our approach can generate a larger class of solutions as (G'/G) are. They arise when more general solutions of the determining system for the polynomials are considered.

Remark 2: In our approach, two successive balancing procedures have been applied:

I. A first balance between the higher derivative and the higher nonlinear terms in the initial equation, following the power of G'. It allowed to determine the value of m in (8) and generated an ODE system in the polynomials $P_{-m}(G),...,P_0(G),...,P_m(G)$.

II. In the determining system for the polynomials $P_i(G)$, a second balancing follow the power of G and it allows to establish the specific form (13) in which the polynomials have to be chosen.

Remark 3: Even by considering the simple KdV model, our approach allowed to get more general solutions as $P_i(G) = a_{-i}G^{-i}$, specific for what G'/G offers.

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Thank you !