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Nonassociative differential geometry and gravity

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based on: Aschieri, Szabo arXiv: 1504.03915;

Aschieri, MDC, Szabo arXiv: 1710.11467.

Noncommutativity & Nonassociativity

NC historically:

-Heisenberg, 1930: regularization of the divergent electron self-energy, coordinates are promoted to noncommuting operators

$$[\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu} \Rightarrow \Delta\hat{x}^\mu \Delta\hat{x}^\nu \geq \frac{1}{2}\Theta^{\mu\nu}.$$

-First model of a NC space-time [Snyder '47].

More recently:

-mathematics: Gelfand-Naimark theorems (C^*-algebras of functions encodes information on topological Hausdorff spaces),
-string theory (open string in a constant B-field),
-new effects in QFT (UV/IR mixing),
-quantum gravity (discretisation of space-time).

NA historically:

-Jordan quantum mechanics [Jordan '32]: hermitean observables do not close an algebra (standard composition, commutator). New composition, $A \circ B$ is hermitean, commutative but nonassociative:

$$A \circ B = \frac{1}{2}((A+B)^2 - A^2 - B^2).$$

-Nambu mechanics: Nambu-Poisson bracket (Poisson bracket) $\{f, g, h\}$ and the fundamental identity (Jacobi identity) [Nambu'73]; quantization still an open problem.

$$f\{g, h, k\} + \{f, h, k\}g = \{fg, h, k\}$$
$$\{f, g, \{h_1, h_2, h_3\}\} + \cdots = 0.$$

More recently:

-mathematics: L_∞ algebras [Stasheff '94; Lada, Stasheff '93].
-string field theory: symmetries of closed string field theory close a strong homotopy Lie-algebra, L_∞ algebra [Zwiebach '15], NA geometry of D-branes in curved backgrounds (NA $\star$-products), closed strings in locally non-geometric backgrounds (low energy limit is a NA gravity).
-magnetic monopoles, NA quantum mechanics.

NC/NA geometry and gravity

Early Universe, singularities of BHs $\Rightarrow$ QG $\Rightarrow$ Quantum space-time
NC/NA space-time $\Longrightarrow$ Gravity on NC/NA spaces.

General Relativity (GR) is based on the diffeomorphism symmetry.
This concept (space-time symmetry) is difficult to generalize to NC/NA spaces. Different approaches:

NC spectral geometry [Chamseddine, Connes, Marcolli '07; Chamseddine, Connes, Mukhanov '14].

Emergent gravity [Steinacker '10, '16].

Frame formalism, operator description [Burić, Madore '14; Fritz, Majid '16].

Twist approach [Wess et al. '05, '06; Ohl, Schenckel '09; Castellani, Aschieri '09; Aschieri, Schenkel '14; Blumenhagen, Fuchs '16; Aschieri, MDC, Szabo, '18].

NC gravity as a gauge theory of Lorentz/Poincaré group [Chamseddine '01,'04, Cardela, Zanon '03, Aschieri, Castellani '09,'12; Dobrski '16].

Overview

NA gravity: General

NA gravity is based on:

-locally non-geometric constant R-flux. Cochain twist $\mathcal{F}$ and associator Φ with

$$\Phi\,(\mathcal{F}\otimes 1)\,(\Delta\otimes\mathrm{id})\mathcal{F} = (1\otimes\mathcal{F})\,(\mathrm{id}\otimes\Delta)\mathcal{F}\ .$$

-equivariance (covariance) under the twisted diffeomorphisms (quasi-Hopf algebra of twisted diffeomorphisms).

-twisted differential geometry in phase space. In particular: connection, curvature, torsion. Projection of phase space (vacuum) Einstein equations to space-time.

-more general, categorical approach in [Barnes, Schenckel, Szabo '14-'16].

Our goals:

-construct NA differential geometry of phase space.

-consistently construct NA deformation of GR in space-time: NA Einstein equations and action; investigate phenomenological consequences.

-understand symmetries of the obtained NA gravity.

NA differntial geometry: Review of twist deformation

Symmetry algebra g and the universal covering algebra Ug.
A well defined way of deforming symmetries: the twist formalism.
Twist $\mathcal{F}$ (introduced by Drinfel'd in 1983-1985) is:
-an invertible element of $Ug \otimes Ug$
-fulfills the 2-cocycle condition (ensures the associativity of the $\star$-product).

$$\mathcal{F} \otimes 1(\Delta \otimes \mathrm{id})\mathcal{F} = 1 \otimes \mathcal{F}(\mathrm{id} \otimes \Delta)\mathcal{F}. \tag{2.1}$$

-additionaly: $\mathcal{F} = 1 \otimes 1 + \mathcal{O}(h)$; h-deformation parameter.

NA differntial geometry: R-flux induced cochain twist

Phase space $\mathcal{M}$: $x^A = (x^\mu, \tilde{x}_\mu = p_\mu)$, $\partial_A = \left(\partial_\mu, \tilde{\partial}^\mu = \frac{\partial}{\partial p_\mu}\right)$.
$2d$ dimensional, $A = 1, \dots 2d$.

The twist $\mathcal{F}$:

$$\mathcal{F} = \exp\left(-\tfrac{\mathrm{i}\,\hbar}{2}\,(\partial_\mu \otimes \tilde{\partial}^\mu - \tilde{\partial}^\mu \otimes \partial_\mu) - \tfrac{\mathrm{i}\,\kappa}{2}\,R^{\mu\nu\rho}\,(p_\nu\,\partial_\rho \otimes \partial_\mu - \partial_\mu \otimes p_\nu\,\partial_\rho)\right)\,, \quad (2.2)$$

with $R^{\mu\nu\rho}$ totally antisymmetric and constant, $\kappa := \frac{\ell_s^3}{6\hbar}$.
Does not fulfill the 2-cocycle condition

$$\Phi\,(\mathcal{F} \otimes 1)\,(\Delta \otimes \mathrm{id})\mathcal{F} = (1 \otimes \mathcal{F})\,(\mathrm{id} \otimes \Delta)\mathcal{F}\ . \quad (2.3)$$

The associator Φ:

$$\Phi = \exp\left(\hbar\kappa\,R^{\mu\nu\rho}\,\partial_\mu \otimes \partial_\nu \otimes \partial_\rho\right) =: \phi_1 \otimes \phi_2 \otimes \phi_3 = 1 \otimes 1 \otimes 1 + O(\hbar\kappa). \quad (2.4)$$

Notation: $\mathcal{F} = \mathrm{f}^\alpha \otimes \mathrm{f}_\alpha$, $\mathcal{F}^{-1} = \bar{\mathrm{f}}^\alpha \otimes \bar{\mathrm{f}}_\alpha$, $\Phi^{-1} =: \bar{\phi}_1 \otimes \bar{\phi}_2 \otimes \bar{\phi}_3$,

Braiding: $\mathcal{R} = \mathcal{F}^{-2} =: \mathrm{R}^{\,\alpha} \otimes \mathrm{R}_{\,\alpha}$, $\mathcal{R}^{-1} = \mathcal{F}^2 =: \overline{\mathrm{R}}^{\,\alpha} \otimes \overline{\mathrm{R}}_{\,\alpha}$.

Hopf aglebra of infinitesimal diffeomorphisms $U\mathrm{Vec}(\mathcal{M})$:

$$[u,v] = (u^B \partial_B v^A - v^B \partial_B u^A)\partial_A,$$
$$\Delta(u) = 1 \otimes u + u \otimes 1,$$
$$\epsilon(u) = 0, S(u) = -u.$$

Quasi-Hopf algebra of infinitesimal diffeomorphisms $U\mathrm{Vec}^{\mathcal{F}}(\mathcal{M})$:

-algebra structure does not change

-coproduct is deformed: $\Delta^{\mathcal{F}}\xi = \mathcal{F}\,\Delta\,\mathcal{F}^{-1}$

-counit and antipod do not change: $\epsilon^{\mathcal{F}} = \epsilon, S^{\mathcal{F}} = S$.

On basis vectors:

$$\begin{aligned} \Delta_{\mathcal{F}}(\partial_\mu) &= 1 \otimes \partial_\mu + \partial_\mu \otimes 1 \ , \\ \Delta_{\mathcal{F}}(\tilde{\partial}^\mu) &= 1 \otimes \tilde{\partial}^\mu + \tilde{\partial}^\mu \otimes 1 + \mathrm{i}\,\kappa\, R^{\mu\nu\rho}\,\partial_\nu \otimes \partial_\rho \ . \end{aligned}$$

NA differential geometry: NA tensor calculus

Guiding principle: Differential geometry on $\mathcal{M}$ is covatiant under $U\mathrm{Vec}(\mathcal{M})$.
NA differential geometry on $\mathcal{M}$ should be covariant under $U\mathrm{Vec}^{\mathcal{F}}(\mathcal{M})$.

In pactice: $U\mathrm{Vec}(\mathcal{M})$-module algebra $\mathcal{A}$ (functions, forms, tensors) and $a, b \in \mathcal{A}$, $u \in \mathrm{Vec}(\mathcal{M})$

$$u(ab) = u(a)b + au(b), \qquad \text{Lie derivative, Leibinz rule (coproduct).}$$

The twist: $U\mathrm{Vec}(\mathcal{M}) \to U\mathrm{Vec}^{\mathcal{F}}(\mathcal{M})$ and $\mathcal{A} \to \mathcal{A}_\star$ with

$$ab \to a \star b = \bar{\mathrm{f}}^{\alpha}(a) \cdot \bar{\mathrm{f}}_{\alpha}(b).$$

Then $\mathcal{A}_\star$ is a $U\mathrm{Vec}^{\mathcal{F}}(\mathcal{M})$-module algebra:

$$\xi(a \star b) = \xi_{(1)}(a) \star \xi_{(2)}(b),$$

for $\xi \in U\mathrm{Vec}^{\mathcal{F}}(\mathcal{M})$ and using the twisted coproduct $\Delta^{\mathcal{F}}\xi = \xi_{(1)} \otimes \xi_{(2)}$.

Commutativity: $a \star b = \bar{\mathrm{f}}^{\,\alpha}(a) \cdot \bar{\mathrm{f}}_{\,\alpha}(b) = \bar{\mathrm{R}}^{\,\alpha}(b) \star \bar{\mathrm{R}}_{\,\alpha}(a) =: {}^{\alpha}b \star_{\alpha} a$

Associativity: $(a \star b) \star c = {}^{\phi_1}a \star ({}^{\phi_2}b \star {}^{\phi_3}c)$.

Functions: $C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})_\star$

$$
\begin{aligned}
f \star g &= \bar{\mathrm{f}}^{\,\alpha}(f) \cdot \bar{\mathrm{f}}_{\,\alpha}(g) \\
&= f \cdot g + \tfrac{\mathrm{i}\,\hbar}{2}\left(\partial_\mu f \cdot \tilde{\partial}^\mu g - \tilde{\partial}^\mu f \cdot \partial_\mu g\right) + \mathrm{i}\,\kappa\, R^{\mu\nu\rho}\, p_\nu\, \partial_\rho f \cdot \partial_\mu g + \cdots \ ,
\end{aligned}
\tag{2.5}
$$

$$[x^\mu \overset{\star}{,} x^\nu] = 2\,\mathrm{i}\,\kappa\, R^{\mu\nu\rho}\, p_\rho, \ [x^\mu \overset{\star}{,} p_\nu] = \mathrm{i}\,\hbar\, \delta^\mu{}_\nu, \ [p_\mu \overset{\star}{,} p_\nu] = 0,$$

$$[x^\mu \overset{\star}{,} x^\nu \overset{\star}{,} x^\rho] = \ell_s^3\, R^{\mu\nu\rho}.$$

NA tensor calculus

Forms: $\Omega^\sharp(\mathcal{M}) \to \Omega^\sharp(\mathcal{M})_\star$

$$\omega \wedge_\star \eta = \overline{\mathrm{f}}^{\,\alpha}(\omega) \wedge \overline{\mathrm{f}}_{\,\alpha}(\eta), \tag{2.6}$$
$$f \star \mathrm{d}x^A = \mathrm{d}x^C \star \left(\delta^A{}_C\, f - \mathrm{i}\,\kappa\,\mathscr{R}^{AB}{}_C\,\partial_B f\right) ,$$

with non-vanishing components $\mathscr{R}^{x^\mu, x^\nu}{}_{\tilde{x}_\rho} = R^{\mu\nu\rho}$. Basis 1-forms

$$(\mathrm{d}x^A \wedge_\star \mathrm{d}x^B) \wedge_\star \mathrm{d}x^C = {}^{\phi_1}(\mathrm{d}x^A) \wedge_\star \left({}^{\phi_2}(\mathrm{d}x^B) \wedge_\star {}^{\phi_3}(\mathrm{d}x^C)\right)$$
$$= \mathrm{d}x^A \wedge_\star (\mathrm{d}x^B \wedge_\star \mathrm{d}x^C) = \mathrm{d}x^A \wedge \mathrm{d}x^B \wedge \mathrm{d}x^C.$$

Exterior derivative d: $\mathrm{d}^2 = 0$ and the undeformed Leibniz rule

$$\mathrm{d}(\omega \wedge_\star \eta) = \mathrm{d}\omega \wedge_\star \eta + (-1)^{|\omega|}\, \omega \wedge_\star \mathrm{d}\eta. \tag{2.7}$$

Duality, $\star$-pairing:

$$\langle\, \omega\,,\, u\,\rangle_\star = \left\langle\, \overline{\mathrm{f}}^{\,\alpha}(\omega)\,,\, \overline{\mathrm{f}}_{\,\alpha}(u)\,\right\rangle . \tag{2.8}$$

NA tensor calculus: Lie derivative

$\star$-Lie drivative:

$$\mathcal{L}^\star_u(T) = \mathcal{L}_{\bar{\mathrm{f}}^{\alpha}(u)}(\bar{\mathrm{f}}_{\alpha}(T)), \tag{2.9}$$

$$\mathcal{L}^\star_u(\omega \wedge_\star \eta) = \mathcal{L}^\star_{\bar{\phi}_1 u}({}^{\bar{\phi}_2}\omega) \wedge_\star {}^{\bar{\phi}_3}\eta + {}^{\alpha}({}^{\bar{\phi}_1\,\bar{\varphi}_1}\omega) \wedge_\star \mathcal{L}^\star_{\alpha({}^{\bar{\phi}_2\,\bar{\varphi}_2}u)}({}^{\bar{\phi}_3\,\bar{\varphi}_3}\eta),$$

$$[\mathcal{L}^\star_u, \mathcal{L}^\star_v]_\bullet = [\bar{\mathrm{f}}^{\alpha}\mathcal{L}^\star_u, \bar{\mathrm{f}}_{\alpha}\mathcal{L}^\star_v] = \mathcal{L}^\star_{[u,v]_\star},$$

with $[u,v]_\star = \left[\bar{\mathrm{f}}^{\alpha}(u), \bar{\mathrm{f}}_{\alpha}(v)\right]$ and

$$\left[u,[v,z]_\star\right]_\star = \left[[{}^{\bar{\phi}_1}u, {}^{\bar{\phi}_2}v]_\star, {}^{\bar{\phi}_3}z\right]_\star + \left[{}^{\alpha}({}^{\bar{\phi}_1\,\bar{\varphi}_1}v), [{}_{\alpha}({}^{\bar{\phi}_2\,\bar{\varphi}_2}u), {}^{\bar{\phi}_3\,\bar{\varphi}_3}z]_\star\right]_\star.$$

Relation of $\mathcal{L}^\star_u$ with diffeomorphism symmetry in space-time needs to be understood.

$\star$-Lie derivative generates "twisted, braided" diffeomorphism symmetry. This symmetry has the L_∞ structure. Work in progress with G. Giotopoulos, V. Radovanović and R. Szabo.

NA differential geometry: connection, torsion, curvature

$\star$-connection:

$$\begin{aligned} \nabla^\star \ : \ \mathsf{Vec}_\star &\longrightarrow \mathsf{Vec}_\star \otimes_\star \Omega^1_\star \\ u &\longmapsto \nabla^\star u \ , \end{aligned} \quad (2.10)$$

$$\nabla^\star(u \star f) = \big(\bar{\phi}_1 \nabla^\star(\bar{\phi}_2 u)\big) \star \bar{\phi}_3 f + u \otimes_\star \mathrm{d} f, \quad (2.11)$$

the right Leibniz rule, for $u \in \mathsf{Vec}_\star$ and $f \in A_\star$. In particular:

$$\nabla^\star \partial_A =: \partial_B \otimes_\star \Gamma^B_A =: \partial_B \otimes_\star (\Gamma^B_{AC} \star \mathrm{d}x^C) \ . \quad (2.12)$$

$$\mathrm{d}_{\nabla^\star}(\partial_A \otimes_\star \omega^A) = \partial_A \otimes_\star (\mathrm{d}\omega^A + \Gamma^A_B \wedge_\star \omega^B),$$

for $\omega^A \in \Omega^\sharp_\star$.

Torsion:

$$\mathsf{T}^\star := \mathrm{d}_{\nabla^\star}\big(\partial_A \otimes_\star \mathrm{d}x^A\big) \ : \ \mathsf{Vec}_\star \otimes_\star \mathsf{Vec}_\star \to \mathsf{Vec}_\star,$$

$$\mathsf{T}^\star(\partial_A, \partial_B) = \partial_C \star (\Gamma^C_{AB} - \Gamma^C_{BA}) =: \partial_C \star \mathsf{T}^C{}_{AB}.$$

Torsion-free condition: $\Gamma^C_{AB} = \Gamma^C_{BA}$.

Curvature:

$$\mathsf{R}^\star := \mathrm{d}_{\nabla^\star} \bullet \mathrm{d}_{\nabla^\star} \ : \ \mathsf{Vec}_\star \longrightarrow \mathsf{Vec}_\star \otimes_\star \Omega^2_\star,$$

$$\mathsf{R}^\star(\partial_A) = \partial_C \otimes_\star (\mathrm{d}\Gamma^C_A + \Gamma^C_B \wedge_\star \Gamma^B_A) = \partial_C \otimes_\star \mathsf{R}^C_A,$$

Ricci tensor:

$$\mathrm{Ric}^\star(u,v) := -\langle\, \mathrm{R}^\star(u,v,\partial_A)\,,\, \mathrm{d}x^A\,\rangle_\star \qquad (2.13)$$
$$\mathrm{Ric}^\star = \mathrm{Ric}_{AD} \star (\mathrm{d}x^D \otimes_\star \mathrm{d}x^A).$$

Commponents from $\mathrm{Ric}_{BC} := \mathrm{Ric}^\star(\partial_B, \partial_C)$

$$\begin{aligned}\mathrm{Ric}_{BC} &= \partial_A\Gamma^A_{BC} - \partial_C\Gamma^A_{BA} + \Gamma^A_{B'A}\star\Gamma^{B'}_{BC} - \Gamma^A_{B'C}\star\Gamma^{B'}_{BA}\\ &+ \mathrm{i}\,\kappa\,\Gamma^A_{B'E}\star\left(\mathscr{R}^{EG}{}_A\,(\partial_G\Gamma^{B'}_{BC}) - \mathscr{R}^{EG}{}_C\,(\partial_G\Gamma^{B'}_{BA})\right) \qquad (2.14)\\ &+ \mathrm{i}\,\kappa\,\mathscr{R}^{EG}{}_A\,\partial_G\partial_C\Gamma^A_{BE} - \mathrm{i}\,\kappa\,\mathscr{R}^{EG}{}_A\,\partial_G\left(\Gamma^A_{B'E}\star\Gamma^{B'}_{BC} - \Gamma^A_{B'C}\star\Gamma^{B'}_{BE}\right)\\ &+ \kappa^2\,\mathscr{R}^{AF}{}_D\left(\mathscr{R}^{EG}{}_A\,\partial_F(\Gamma^D_{B'E}\star\partial_G\Gamma^{B'}_{BC}) - \mathscr{R}^{EG}{}_C\,\partial_F(\Gamma^D_{B'E}\star\partial_G\Gamma^{B'}_{BA})\right)\ .\end{aligned}$$

Scalar curvature cannot be defined along these lines: cannot be seen as a map and inverse metric tensor needed. Not straightforward:

$$G^{MN}\star G_{NP} = \delta^P_M, \ \text{ but } \ (G^{MN}\star G_{NP})\star f \neq G^{MN}\star(G_{NP}\star f).$$

NA deformation of GR: NA Levi-Civita connection

GR connection $\Gamma^{\mathrm{LC}\,\rho}_{\mu\nu}$ is a Levi-Civita connection: torssion-free and metric compatible $\nabla_\alpha g_{\mu\nu} = 0$.

Generalization: $\mathrm{g}^\star \in \Omega^1_\star \otimes_\star \Omega^1_\star$ and ${}^\star\nabla \mathrm{g}^\star = 0$.

Connection coefficients, expanded up to first order in $\hbar\kappa$:

$$\begin{aligned}
\Gamma^{S(0,0)}_{AD} &= \Gamma^{\mathrm{LC}\,S}_{AD} = \tfrac{1}{2}\,\mathrm{g}^{SQ}\left(\partial_D \mathrm{g}_{AQ} + \partial_A \mathrm{g}_{DQ} - \partial_Q \mathrm{g}_{AD}\right), \qquad (3.15)\\
\Gamma^{S(0,1)}_{AD} &= -\frac{\mathrm{i}\,\hbar}{2}\,\mathrm{g}^{SP}\left(\left(\partial_\mu \mathrm{g}_{PQ}\right)\tilde{\partial}^\mu \Gamma^{\mathrm{LC}\,Q}_{AD} - \left(\tilde{\partial}^\mu \mathrm{g}_{PQ}\right)\partial_\mu \Gamma^{\mathrm{LC}\,Q}_{AD}\right),\\
\Gamma^{S(1,0)}_{AD} &= \mathrm{i}\,\kappa\,R^{\alpha\beta\gamma}\left(\tilde{\mathrm{g}}^S{}_\gamma\,\mathrm{g}_{\beta N}\left(\partial_\alpha \Gamma^{\mathrm{LC}\,N}_{AD}\right) - \mathrm{g}^{SM}\,p_\beta\left(\partial_\gamma \mathrm{g}_{MN}\right)\partial_\alpha \Gamma^{\mathrm{LC}\,N}_{AD}\right),\\
\Gamma^{S(1,1)}_{AD} &= \frac{\hbar\,\kappa}{2}\,R^{\alpha\beta\gamma}\left[\ \text{long expression } + \left(\partial_\alpha \mathrm{g}^{SQ}\right)\left(\partial_\beta \mathrm{g}_{QP}\right)\partial_\gamma \Gamma^{\mathrm{LC}\,P}_{AD}\right].
\end{aligned}$$

Comments:

-$\Gamma_{AD}^{S(0,1)}$ and $\Gamma_{AD}^{S(1,0)}$ imaginary, $\Gamma_{AD}^{S(1,1)}$ real.

-for g_{MN} that does not depend on the momenta p_μ, only the last term in $\Gamma_{AD}^{S(1,1)}$ remains.

-$\tilde{g}^S_{\ \gamma} = g^{SM}\,\delta_{M,\tilde{x}_\gamma}$.

NA deformation of GR: NA vacuum Einstein equation

We can write vacuum Einstein equations in phase space as:

$$\mathrm{Ric}_{BC} = 0 \ . \tag{3.16}$$

Our strategy: expand Ricci tensor (2.13) in term of (3.15), i. e. the metric tensor g_{MN}. This gives Einstein equations in phase space. How do we obtain the induced equations in space-time?

- start from objects in space-time M $g = g_{\mu\nu}\mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu}$ and lift them to phase space $\mathcal{M}$ foliated with leaves of constant momenta, each leave is diffeomorphic to M.

$$\begin{array}{ccc}
C^{\infty}(\mathcal{M}) & \xrightarrow{\ Q\ } & \widehat{C^{\infty}(\mathcal{M})} \\
\uparrow {\scriptstyle \pi^*} & & \downarrow {\scriptstyle s_{\bar{p}}^* = \sigma^*} \\
C^{\infty}(M) & \xrightarrow[Q_{\bar{p}}]{\cdots\cdots} & \widehat{C^{\infty}(M)}
\end{array}$$

Metric tensor: $g = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \to \hat{\mathrm{g}}_{MN}\,\mathrm{d}x^M \otimes \mathrm{d}x^N$ with

$$\left(\hat{\mathrm{g}}_{MN}(x)\right) = \begin{pmatrix} \mathrm{g}_{\mu\nu}(x) & 0 \\ 0 & \mathrm{h}^{\mu\nu}(x) \end{pmatrix} . \qquad (3.17)$$

Note the additional nondegenerate bilinear $\mathrm{h}(x)^{\mu\nu}\,\mathrm{d}\tilde{x}_\mu \otimes \mathrm{d}\tilde{x}_\nu$; natural choice $\mathrm{h}(x)^{\mu\nu} = \eta^{\mu\nu}$.

- Do all calculations in phase space, using the twisted differential geometry. In particular, calculate Ric_{BC} in terms of g_{AB}, (2.13), (3.15).
- Finally, project the result to space-time using the zero section $x \mapsto \sigma(x) = (x, 0)$.

  Functions, forms: pullback to the zero momentum leaf:

  Vector fields: $v^\mu(x,p)\,\partial_\mu + \tilde{v}_\mu(x,p)\,\tilde{\partial}^\mu \mapsto v^\mu(x,0)\,\partial_\mu$.

Ricci tensor: $\mathrm{Ric} \to \mathrm{Ric}^{\star\circ} = \mathrm{Ric}^\circ_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$,
$\mathrm{Ric}^\circ_{\mu\nu}(x) = \sigma^*(\mathrm{Ric}_{\mu\nu})(x,p) = \mathrm{Ric}_{\mu\nu}(x,0)$.

NA deformation of GR: NA gravity in space-time

The lifted metric $\hat{g}_{MN}\, \mathrm{d}x^M \otimes \mathrm{d}x^N = g_{MN} \star (\mathrm{d}x^M \otimes_\star \mathrm{d}x^N)$,

$$g_{MN}(x) = \begin{pmatrix} g_{\mu\nu}(x) & \frac{\mathrm{i}\,\kappa}{2}\, R^{\sigma\nu\alpha}\,\partial_\sigma g_{\mu\alpha} \\ \frac{\mathrm{i}\,\kappa}{2}\, R^{\sigma\mu\alpha}\,\partial_\sigma g_{\alpha\nu} & \eta^{\mu\nu}(x) \end{pmatrix} . \tag{3.18}$$

Ricci tensor in space-time, (expanded up to first order in $\hbar\kappa$):

$$\begin{aligned} \mathrm{Ric}^{\circ}_{\mu\nu} = \mathrm{Ric}^{\mathrm{LC}}_{\mu\nu} + \tfrac{\ell_s^3}{12}\, R^{\alpha\beta\gamma} \Big(& \partial_\rho \big(\partial_\alpha g^{\rho\sigma}\, (\partial_\beta g_{\sigma\tau})\, \partial_\gamma \Gamma^{\mathrm{LC}\,\tau}_{\mu\nu}\big) \\ & -\partial_\nu \big(\partial_\alpha g^{\rho\sigma}\, (\partial_\beta g_{\sigma\tau})\, \partial_\gamma \Gamma^{\mathrm{LC}\,\tau}_{\mu\rho}\big) \\ + \partial_\gamma g_{\tau\omega} \big(& \partial_\alpha (g^{\sigma\tau}\, \Gamma^{\mathrm{LC}\,\rho}_{\sigma\nu})\, \partial_\beta \Gamma^{\mathrm{LC}\,\omega}_{\mu\rho} - \partial_\alpha (g^{\sigma\tau}\, \Gamma^{\mathrm{LC}\,\rho}_{\sigma\rho})\, \partial_\beta \Gamma^{\mathrm{LC}\,\omega}_{\mu\nu} \\ + (\Gamma^{\mathrm{LC}\,\sigma}_{\mu\rho}\, & \partial_\alpha g^{\rho\tau} - \partial_\alpha \Gamma^{\mathrm{LC}\,\sigma}_{\mu\rho}\, g^{\rho\tau})\, \partial_\beta \Gamma^{\mathrm{LC}\,\omega}_{\sigma\nu} \\ - (\Gamma^{\mathrm{LC}\,\sigma}_{\mu\nu}\, & \partial_\alpha g^{\rho\tau} - \partial_\alpha \Gamma^{\mathrm{LC}\,\sigma}_{\mu\nu}\, g^{\rho\tau})\, \partial_\beta \Gamma^{\mathrm{LC}\,\omega}_{\sigma\rho} \big) \Big). \end{aligned} \tag{3.19}$$

Vacuum Einstein equations in space-time:

$$\mathrm{Ric}^{\circ}_{\mu\nu} = 0. \tag{3.20}$$

NA deformation of GR: Comments

- R-flux (via NA differential geometry) generates non-trivial dynamical consequences on spacetime, they are independent of $\hbar$ and real-valued.
- Why zero momentum leaf? Pulling back to a leaf of constant momentum $p = p^\circ$ (generally) gives a non-vanishing imaginary contribution $\mathrm{Ric}^{(1,0)}_{\mu\nu}\big|_{p=p^\circ}$ to the spacetime Ricci tensor. Also, n-triproducts calculated on the zero momentum leaf [Aschieri, Szabo '15] coincide with those proposed in [Munich group '11].
- Why $\mathrm{h}(x)^{\mu\nu} = \eta^{\mu\nu}$? The simplest choice, can be extended. In relation with Born geometry [Freidel et al. '14]: in our model nonassociativity does not generates curved momentum space. Investigate $\mathrm{h}(x)^{\mu\nu} \neq \eta^{\mu\nu} \ldots$

Discussion

Our goals:

- Phenomenological consequences (R-flux induced corrections to GR solutions): to be investigates.
- Construction of scalar curvature, matter fields, full Einstein equations: to be investigated.
- Twisted diffeomorphism symmetry: to be understood better, L_∞ structure?
- NA gravity as a gauge theory of Lorentz symmetry, NA Einstein-Cartan gravity: better understanding of NA gauge symmetry is needed, L_∞ structure?

Introduce:

$$F_Q = \exp\big(-\tfrac{\mathrm{i}\,\kappa}{2}\,Q^{\mu\nu}{}_{\rho}\,(w^\rho\,\partial_\mu\otimes\partial_\nu-\partial_\nu\otimes w^\rho\,\partial_\mu)\big) \tag{4.21}$$

and

$$\hat F = \exp\big(-\tfrac{\mathrm{i}\,\hbar}{2}\,(\hat\partial^\mu\otimes\hat{\tilde\partial}_\mu-\hat{\tilde\partial}_\mu\otimes\hat\partial^\mu)\big) \tag{4.22}$$

with w^μ closed string winding coordinates, regard it as momenta $\hat p^\mu$ conjugate to coordinates $\hat x_\mu$ T-dual to the spacetime variables x^μ. Then the twist element in the full phase space $\mathcal{M}\times\hat{\mathcal{M}}$ of double field theory in the R-flux frame is:

$$\hat{\mathcal{F}} = \mathcal{F}\,F_Q\,\hat F\ . \tag{4.23}$$

$O(2d,2d)$-invariant twist; can be rotated to any other T-duality frame by using an $O(2d,2d)$ transformation on $\mathcal{M}\times\hat{\mathcal{M}}$. A nonassociative theory which is manifestly invariant under $O(2d,2d)$ rotations.