Courant and Roytenberg bracket and their relation via T-duality

Ilija Ivanišević

Overview

1. Bosonic string
2. Symmetry generators and generalized currents
3. Courant, twisted Courant and Roytenberg brackets

- This presentation is mostly based on the paper arXiv:1903.04832 by Lj. Davidovic, I. Ivanisevic, B. Sazdovic

Action

- Action for closed bosonic string moving in the coordinate dependant background:

$$S[x] = \kappa \int_\Sigma d^2\xi \sqrt{-g}\Big[\big(\frac{1}{2} g^{\alpha\beta} G_{\mu\nu}(x) + \frac{\epsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu}(x)\big)\partial_\alpha x^\mu \partial_\beta x^\nu + \Phi(x) R^{(2)}\Big]$$

- If the dilaton field is taken to be zero, action can be rewritten using the light-cone coordinate system $\xi^\pm = \frac{1}{2}(\tau \pm \sigma)$, $\partial_\pm = \partial_0 \pm \partial_1$ in conformal gauge $g_{\alpha\beta} = e^F \eta_{\alpha\beta}$:

$$S[x] = \kappa \int_\Sigma d^2\xi \partial_+ x^\mu \Pi_{+\mu\nu}[x] \partial_- x^\nu, \quad \Pi_{\pm\mu\nu}[x] = B_{\mu\nu}[x] \pm \frac{1}{2} G_{\mu\nu}[x]$$

Hamiltonian

- Canonical momenta:

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \kappa G_{\mu\nu}(x)\dot{x}^\nu - 2\kappa B_{\mu\nu}(x)x'^\nu$$

- Hamiltonian is the Legendre transform of the Lagrangian

$$\mathcal{H}_\mathcal{C} = \frac{1}{2\kappa}\pi_\mu (G^{-1})^{\mu\nu}\pi_\nu - 2x'^\mu B_{\mu\nu}(G^{-1})^{\nu\rho}\pi_\rho + \frac{\kappa}{2}x'^\mu G^E_{\mu\nu}x'^\nu,$$

where $G^E_{\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}$ is the effective metric.

- Hamiltonian can be rewriten as a function of some currents $j_{\pm\mu}$:

$$\mathcal{H}_\mathcal{C} = \frac{1}{4\kappa}(G^{-1})^{\mu\nu}\left[j_{+\mu}j_{+\nu} + j_{-\mu}j_{-\nu}\right],$$

$$j_{\pm\mu}(x) = \pi_\mu + 2\kappa\Pi_{\pm\mu\nu}(x)x'^\nu$$

- Currents can be rewritten in the following form:

$$j_{\pm\mu} = i_\mu \pm \kappa G_{\mu\nu}x'^\nu, \quad i_\mu = \pi_\mu + 2\kappa B_{\mu\nu}x'^\nu$$

T-DUALITY

- T-duality is an equivalence of two seemingly different physical theories in a way that all observable quantities in one theory are identified with quantities in its dual theory.
- Duality transformations are not symmetry transformations - action is not invariant.
- Example: closed bosonic string with one dimension being compactified to a circle
- Mass spectrum:
$$M^2 = \frac{K^2}{R^2} + W^2 \frac{R^2}{\alpha'^2},$$
- Spectrum remains invariant under exchange $K \leftrightarrow W$ and $R \leftrightarrow \frac{\alpha'}{R}$
- Momenta in one theory are winding numbers in its T-dual theory and vice versa.

T-duality

- Coordinates and momenta relation:

$$\pi_\mu \simeq \kappa x'^\mu$$

- T-duality transformation laws for background fields:

$$^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}, \quad ^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}$$

$\theta^{\mu\nu} = -\frac{2}{\kappa}(G_E^{-1} B G^{-1})^{\mu\nu}$ is the non-commutativity parameter.

- Currents transformation under T-duality:

$$i_\mu \simeq \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu \equiv k^\mu,$$
$$j_{\pm\mu} \simeq k^\mu \pm (G_E^{-1})^{\mu\nu}\pi_\nu \equiv {}^\star j_\mp^\mu$$

Generalized currents

- So far we have introduced two types of currents:

$$j_{\pm\mu} = i_\mu \pm G_{\mu\nu}\kappa x'^\nu, \quad \mathcal{H}_\mathcal{C} = \frac{1}{4\kappa}(G^{-1})^{\mu\nu}\Big[j_{+\mu}j_{+\nu} + j_{-\mu}j_{-\nu}\Big],$$
$$^\star j_\pm^\mu = k^\mu \pm (G_E^{-1})^{\mu\nu}\pi_\nu, \quad \mathcal{H}_\mathcal{C} = \frac{1}{4\kappa}G^E_{\mu\nu}\Big[{}^\star j_+^\mu {}^\star j_+^\nu + {}^\star j_-^\mu {}^\star j_-^\nu\Big]$$

- Consider two sets of generalized currents:

$$J_C(\xi, \Lambda^C) = \xi^\mu(x) i_\mu + \Lambda^C_\mu(x)\kappa x'^\mu,$$
$$J_R(\xi_R, \Lambda) = \xi^\mu_R(x)\pi_\mu + \Lambda_\mu(x) k^\mu.$$

- These generalized currents are mutually T-dual

$$i_\mu \simeq k^\mu, \; \kappa x'^\mu \simeq \pi_\mu \longrightarrow J_C(\xi, \Lambda^C) \simeq J_R(\xi_R, \Lambda)$$

Generalized currents

- Aforementioned generalized currents are special cases of

$$J(\xi, \Lambda) = \xi^{\mu} \pi_{\mu} + \Lambda_{\mu} \kappa x'^{\mu}.$$

$$\Lambda^{C}_{\mu} = \Lambda_{\mu} + 2B_{\mu\nu}\xi^{\nu}, \quad J(\xi, \Lambda) \to J_C(\xi, \Lambda^C)$$

$$\xi^{\mu}_{R} = \xi^{\mu} + \kappa\theta^{\mu\nu}\Lambda_{\nu}, \quad J(\xi, \Lambda) \to J_R(\xi_R, \Lambda)$$

- This current is self T-dual.
- It is related to symmetry generators.

General coordinate transformations

- Action of general coordinate transformations on background fields:

$$\begin{aligned} \delta_\xi G_{\mu\nu} &= \mathcal{L}_\xi G_{\mu\nu}, \\ \delta_\xi B_{\mu\nu} &= \mathcal{L}_\xi B_{\mu\nu}, \end{aligned}$$

  where Lie derivative $\mathcal{L}_\xi = i_\xi d + d i_\xi$ represents the change of a tensor field along the flow defined by the vector field ξ.
- General coordinate transformations are generated by

$$\mathcal{G}_{GCT}(\xi) = \int_0^{2\pi} d\sigma \xi^\mu(x) \pi_\mu \,.$$

- The Poisson bracket algebra of generators gives rise to the Lie bracket

$$\{\mathcal{G}_{GCT}(\xi_1), \mathcal{G}_{GCT}(\xi_2)\} = -\mathcal{G}_{GCT}([\xi_1, \xi_2]_L).$$

- Lie bracket is defined as the commutator of Lie derivatives

$$[\xi_1, \xi_2]_L = \mathcal{L}_{\xi_1}\mathcal{L}_{\xi_2} - \mathcal{L}_{\xi_2}\mathcal{L}_{\xi_1}.$$

Local gauge transformations

- Action of local gauge transformations on background fields:

$$\delta_\Lambda G_{\mu\nu} = 0,$$
$$\delta_\Lambda B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$$

- Generator of local gauge transformation:

$$\mathcal{G}_{LGT}(\Lambda) = \int d\sigma \Lambda_\mu \kappa x'^\mu .$$

- Generator of general coordinate transformations and local gauge transformations:

$$\mathcal{G}(\xi, \Lambda) = \int d\sigma \left[\xi^\mu \pi_\mu + \Lambda_\mu \kappa x'^\mu \right] = \int d\sigma (\Lambda^T)^M \Omega_{MN} X^N .$$

$$\Lambda^M = \begin{pmatrix} \xi^\mu \\ \Lambda_\mu \end{pmatrix}, \; X^M = \begin{pmatrix} \kappa x'^\mu \\ \pi_\mu \end{pmatrix}, \Omega_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

- Generator $\mathcal{G}(\xi, \Lambda)$ is self T-dual.

Courant bracket

- Poisson bracket algebra:

$$\{\mathcal{G}(\xi_1,\Lambda_1),\,\mathcal{G}(\xi_2,\Lambda_2)\} = \mathcal{G}(\xi,\Lambda)$$

$$\begin{aligned}
\xi^\mu &= \xi_2^\nu \partial_\nu \xi_1^\mu - \xi_1^\nu \partial_\nu \xi_2^\mu \\
\Lambda_\mu &= \xi_2^\nu(\partial_\nu \Lambda_{1\mu} - \partial_\mu \Lambda_{1\nu}) - \xi_1^\nu(\partial_\nu \Lambda_{2\mu} - \partial_\mu \Lambda_{2\nu})
\end{aligned}$$

- The above relation can be rewritten in the following form:

$$\{\mathcal{G}(\xi_1,\Lambda_1),\,\mathcal{G}(\xi_2,\Lambda_2)\} = -\mathcal{G}([\xi_1+\Lambda_1,\xi_2+\Lambda_2]_C),$$

- Generators algebra gives rise to the Courant bracket in the same way that general coordinate transformations generators algebra gives rise to the Lie bracket.
- Courant bracket is the operation on the direct sum of the tangent bundle and the vector bundle of 1-forms.

Courant bracket

- Courant bracket:

$$[\xi_1 + \Lambda_1, \xi_2 + \Lambda_2]_C = [\xi_1, \xi_2]_L + \mathcal{L}_{\xi_1}\Lambda_2 - \mathcal{L}_{\xi_2}\Lambda_1 - \frac{1}{2} d(i_{\xi_1}\Lambda_2 - i_{\xi_2}\Lambda_1)$$

- Courant bracket is not Lie bracket, since it does not satisfy the Jacobi identity.

$$[[\xi_1 + \Lambda_1, \xi_2 + \Lambda_2]_C, \xi_3 + \Lambda_3]_C + cyclic = \mathrm{d}Nij(\xi_1 + \Lambda_1, \xi_2 + \Lambda_2, \xi_3 + \Lambda_3)_C$$

$$Nij(\xi_1 + \Lambda_1, \xi_2 + \Lambda_2, \xi_3 + \Lambda_3)_C = \frac{1}{3} < (\xi_1 + \Lambda_1, \xi_2 + \Lambda_2)_C, \xi_3 + \Lambda_3 > +cyclic$$

$$< \xi_1 + \Lambda_1, \xi_2 + \Lambda_2 >= \frac{1}{2}(\xi_1(\Lambda_2) - \xi_2(\Lambda_1))$$

Change of parameter

- Change of parameter: $\Lambda^C_\mu = \Lambda_\mu + 2B_{\mu\nu}\xi^\nu$.
- This correspond to B-transformation:

$$e^{\hat{B}} = \begin{pmatrix} 1 & 0 \\ 2B & 1 \end{pmatrix}, \ \Lambda^M \to (e^{\hat{B}})^M{}_N \Lambda^N$$

- Generator can be rewritten with new parameters in a new basis

$$\mathcal{G}_{\mathcal{C}}(\xi, \Lambda^C) = \int d\sigma \Big[\xi^\mu i_\mu + \kappa \Lambda^C_\mu x'^\mu\Big], \ i_\mu = \pi_\mu + 2\kappa B_{\mu\nu} x'^\nu$$

- Generator $\mathcal{G}_{\mathcal{C}}$ can be seen as a charge corresponding to the generalized current $J_C(\xi, \Lambda^C) = \xi^\mu i_\mu + \Lambda^C_\mu \kappa x'^\mu$.

Twisted Courant bracket

- H-flux:

$$\{i_\mu(\sigma), i_\nu(\bar\sigma)\} = -2\kappa B_{\mu\nu\rho} x'^\rho \delta(\sigma - \bar\sigma),$$

where $B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ is Kalb-Ramond field strength.

- Poisson bracket algebra:

$$\{\mathcal{G}_\mathcal{C}(\xi_1, \Lambda_1^C), \mathcal{G}_\mathcal{C}(\xi_2, \Lambda_2^C)\} = \mathcal{G}_\mathcal{C}(\xi, \Lambda^C)$$

$$\xi^\mu = \xi_2^\nu \partial_\nu \xi_1^\mu - \xi_1^\nu \partial_\nu \xi_2^\mu$$

$$\Lambda_\mu = \xi_2^\nu(\partial_\nu \Lambda_{1\mu}^C - \partial_\mu \Lambda_{1\nu}^C) - \xi_1^\nu(\partial_\nu \Lambda_{2\mu}^C - \partial_\mu \Lambda_{2\nu}^C) - 2\kappa B_{\mu\nu\rho}\xi_1^\nu \xi_2^\rho$$

- The above relations can be rewritten in the following way:

$$\{\mathcal{G}_\mathcal{C}(\xi_1, \Lambda_1^C), \mathcal{G}_\mathcal{C}(\xi_2, \Lambda_2^C)\} = -\mathcal{G}_\mathcal{C}([\xi_1 + \Lambda_1^C, \xi_2 + \Lambda_2^C]_B).$$

- Change in bracket corresponds to the twisting of the Courant bracket by $2B_{\mu\nu}$. The difference between twisted and untwisted bracket:

$$[e^B(\xi_1 + \Lambda_1), e^B(\xi_2 + \Lambda_2)]_C - e^B[\xi_1 + \Lambda_1, \xi_2 + \Lambda_2]_C = i_{\xi_1} i_{\xi_2} B$$

- Expression for twisted Courant bracket:

$$[\xi_1+\Lambda_1^C, \xi_2+\Lambda_2^C]_B = [\xi_1, \xi_2]_L + \mathcal{L}_{\xi_1}\Lambda_2^C - \mathcal{L}_{\xi_2}\Lambda_1^C - \frac{1}{2}d(i_{\xi_1}\Lambda_2^C - i_{\xi_2}\Lambda_1^C) + H(\xi_1, \xi_2, .), \; H = 2dB$$

Change of parameter

- Change of parameter: $\xi_R^\mu = \xi^\mu + \kappa\theta^{\mu\nu}\Lambda_\nu$
- This correspond to β-transformation for $\beta = \kappa\theta$:

$$e^{\hat{\beta}} = \begin{pmatrix} 1 & \kappa\theta \\ 0 & 1 \end{pmatrix}, \ \Lambda^M \to (e^{\hat{\beta}})^M{}_N \Lambda^N$$

- Generator can be rewritten with new parameters in a new basis

$$\mathcal{G}_{\mathcal{R}}(\xi_R, \Lambda) = \int d\sigma \Big[\xi_R^\mu \pi_\mu + \Lambda_\mu k^\mu\Big], \ k^\mu = \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu.$$

- T-duality relation: $\mathcal{G}_{\mathcal{R}}(\xi, \Lambda^C) \simeq \mathcal{G}_{\mathcal{C}}(\xi_R, \Lambda)$.
- The generator $\mathcal{G}_{\mathcal{R}}$ can be seen as the charge corresponding to generalized current $J_R(\xi_R, \Lambda) = \xi_R^\mu \pi_\mu + \Lambda_\mu k^\mu$.

Roytenberg bracket

- Q and R flux:

$$\{k^\mu(\sigma), k^\nu(\bar\sigma)\} = -\kappa Q_\rho{}^{\mu\nu} k^\rho \delta(\sigma-\bar\sigma) - \kappa^2 R^{\mu\nu\rho}\pi_\rho \delta(\sigma-\bar\sigma),$$

  where $Q_\rho{}^{\mu\nu} = \partial_\rho \theta^{\mu\nu}$, $R^{\mu\nu\rho} = \theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma\theta^{\rho\mu} + \theta^{\rho\sigma}\partial_\sigma\theta^{\mu\nu}$.
- Generators algebra give rise to the Roytenberg bracket:

$$\{\mathcal{G}_{\mathcal{R}}(\xi_1^R, \Lambda_1), \mathcal{G}_{\mathcal{R}}(\xi_2^R, \Lambda_2)\} = -\mathcal{G}_{\mathcal{R}}([\xi_1^R + \Lambda_1, \xi_2^R + \Lambda_2]_R),$$

- Roytenberg bracket is a generalization of Courant bracket, obtained by twisting the Courant bracket by some bi-vector Π. It differs from the Courant bracket by a following term:

$$[e^\Pi(\xi_1+\Lambda_1), e^\Pi(\xi_2+\Lambda_2)]_C - e^\Pi[\xi_1+\Lambda_1, \xi_2+\Lambda_2]_C$$

- Poisson bracket algebra gives rise to the Roytenberg bracket, obtained by twisting the Courant bracket by a bi-vector $\Pi^{\mu\nu} = \kappa\theta^{\mu\nu}$.

Roytenberg bracket

- Roytenberg bracket:

$$
\begin{aligned}
[\xi_1+\Lambda_1,\xi_2+\Lambda_2]_R =&[\xi_1,\xi_2]_L + L_{\xi_1}\Lambda_2 - L_{\xi 2}\Lambda_1 - \frac{1}{2}d(i_{\xi 1}\Lambda_2 - i_{\xi 2}\Lambda_1) - \\
&H\Pi(\xi_1,\xi_2) + \Pi H(\Lambda_1,\xi_2,.) - \Pi H(\Lambda_2,\xi_1,.) \\
&(L_{\xi 2}\Lambda_1 - L_{\xi_1}\Lambda_2 + \frac{1}{2}d(i_{\xi 1}\Lambda_2 - i_{\xi 2}\Lambda_1))\Pi + \\
&\Lambda^2\Pi H(\Lambda_1,.,\xi_2) - \Lambda^2\Pi H(\Lambda_2,.,\xi_1) - [\Lambda_1,\Lambda_2]_\Pi + \\
&\Lambda^2\Pi H(\Lambda_1,\Lambda_2,.) - [\xi_2,\Lambda_1\Pi]_L + [\xi_1,\Lambda_2\Pi]_L + \\
&\left(\frac{1}{2}[\Pi,\Pi]_S - \Lambda^3\Pi H\right)(\Lambda_1,\Lambda_2,.) + H(\xi_1,\xi_2,.)
\end{aligned}
$$

  - Koszul bracket is a generalization of the Lie bracket on the space of differential forms $[\xi,\eta]_\Pi = \mathcal{L}_{\Pi\xi}\eta - \mathcal{L}_{\Pi\eta}\xi + d(\Pi(\xi,\eta))$
  - Schouten-Nijenhuis bracket is a generalization of the Lie bracket on the space of multivectors: $[\Pi,\Pi]_S|^{\mu\nu\rho} = \epsilon^{\mu\nu\rho}_{\alpha\beta\gamma}\Pi^{\sigma\alpha}\partial_\sigma\Pi^{\beta\gamma}$

Questions?