The Elliptic Gaudin Model with Boundary

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Outline

Elliptic Gaudin Model

- The elliptic model has interesting algebraic, geometrical and functional structures.
- Both the rational and the trigonometric models can be obtained as appropriate limits of the elliptic one.
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R-matrix of the XYZ chain

The R-matrix of the XYZ chain is given by

$$R(\lambda, \eta, \kappa) = \mathbb{1} + \sum_{\alpha=1}^{3} W_\alpha(\lambda, \eta, \kappa)\ \sigma^\alpha \otimes \sigma^\alpha,$$

were we use $\mathbb{1}$ for the identity matrix,

$$W_1(\lambda, \eta, \kappa) = \frac{\mathrm{cn}(\lambda + \eta, \kappa)\ \mathrm{sn}(\eta, \kappa)}{\mathrm{sn}(\lambda + \eta, \kappa)\ \mathrm{cn}(\eta, \kappa)}, W_2(\lambda, \eta, \kappa) = \frac{\mathrm{dn}(\lambda + \eta, \kappa)\ \mathrm{sn}(\eta, \kappa)}{\mathrm{sn}(\lambda + \eta, \kappa)\ \mathrm{dn}(\eta, \kappa)},$$

$$W_3(\lambda, \eta, \kappa) = \frac{\mathrm{sn}(\eta, \kappa)}{\mathrm{sn}(\lambda + \eta, \kappa)},$$

the functions $\mathrm{sn}(\lambda, \kappa)$, $\mathrm{cn}(\lambda, \kappa)$, and $\mathrm{dn}(\lambda, \kappa)$ are the usual Jacobi elliptic functions, λ is a spectral parameter, η is a quasi-classical parameter, κ is the modulus and

R-matrix of the XYZ chain

σ^α, $\alpha = 1, 2, 3$, are the Pauli matrices

$$\sigma^\alpha = \begin{pmatrix} \delta_{\alpha 3} & \delta_{\alpha 1} - \imath \delta_{\alpha 2} \\ \delta_{\alpha 1} + \imath \delta_{\alpha 2} & -\delta_{\alpha 3} \end{pmatrix}.$$

This R-matrix satisfies the Yang-Baxter equation

$$R_{12}(\lambda - \mu) R_{13}(\lambda) R_{23}(\mu) = R_{23}(\mu) R_{13}(\lambda) R_{12}(\lambda - \mu).$$

In the present case Yang-Baxter equation reduces to the following matrix equation

$$\sum_{\alpha,\beta,\gamma=1}^{3} \epsilon_{\alpha\beta\gamma} \left(W_\beta(\lambda - \mu) W_\gamma(\lambda) - W_\alpha(\lambda - \mu) W_\gamma(\mu) + W_\alpha(\lambda) W_\beta(\mu) \right.$$

$$\left. - W_\gamma(\lambda - \mu) W_\beta(\lambda) W_\alpha(\mu) \right) \sigma^\alpha \otimes \sigma^\beta \otimes \sigma^\gamma = 0.$$

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Some Properties of the R-matrix

- unitarity $R(\lambda)R(-\lambda) = \rho(\lambda, \eta, \kappa)\ \mathbb{1}$, where the function $\rho(\lambda, \eta, \kappa)$ is given by

$$\rho(\lambda, \eta, \kappa) = \left(1 + \sum_{\alpha=1}^{3} W_\alpha(\lambda) W_\alpha(-\lambda)\right)$$

$$= 4 \frac{\mathrm{sn}^2(\eta, \kappa)}{\mathrm{sn}^2(2\eta, \kappa)} \frac{\mathrm{sn}^2(\lambda, \kappa) - \mathrm{sn}^2(2\eta, \kappa)}{\mathrm{sn}^2(\lambda, \kappa) - \mathrm{sn}^2(\eta, \kappa)}.$$

- parity invariance $R_{21}(\lambda) = R_{12}(\lambda)$;
- temporal invariance $R_{12}^{t}(\lambda) = R_{12}(\lambda)$;
- crossing symmetry $R(\lambda) = \mathcal{J}_1 R^{t_1}(-\lambda - 2\eta)\mathcal{J}_1$,
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Hilbert Space

We study an inhomogeneous XYZ spin chain with N sites, with the local space V_m, that is the $2s+1$ dimensional spin s representation space of the Sklyanin algebra and inhomogeneous parameter α_j.

$$\mathcal{H} = \bigotimes_{m=1}^{N} V_m .$$

Lax Operator

Following Sklyanin we introduce the Lax operator

$$\mathbb{L}_{0q}(\lambda) = \mathbb{1} \otimes S^0 + \sum_{\alpha=1}^{3} W_\alpha(\lambda,\eta,\kappa)\ \sigma^\alpha \otimes S^\alpha,$$
$$= \begin{pmatrix} S^0 + W_3(\lambda)S^3 & W_1(\lambda)S^1 - \imath W_2(\lambda)S^2 \\ W_1(\lambda)S^1 + \imath W_2(\lambda)S^2 & S^0 - W_3(\lambda)S^3 \end{pmatrix},$$

were S^0, S^1, S^2, S^3 are the generators of the Sklyanin algebra $U_{\tau,\eta}(sl(2))$.

Sklyanin Algebra

The generators of the Sklyanin algebra satisfy the following relations

$$\begin{aligned}
\left[S^1, S^2\right] &= \imath \left(S^0 S^3 + S^3 S^0\right), \\
\left[S^2, S^3\right] &= \imath \left(S^0 S^1 + S^1 S^0\right), \\
\left[S^3, S^1\right] &= \imath \left(S^0 S^2 + S^2 S^0\right), \\
\left[S^0, S^1\right] &= \imath J_{23} \left(S^2 S^3 + S^3 S^2\right), \\
\left[S^0, S^2\right] &= \imath J_{31} \left(S^3 S^1 + S^1 S^3\right), \\
\left[S^0, S^3\right] &= \imath J_{12} \left(S^1 S^2 + S^2 S^1\right),
\end{aligned}$$

Sklyanin Algebra

where

$$J_{23} = \frac{W_2(\lambda-\mu)W_3(\lambda)W_2(\mu) - W_3(\lambda-\mu)W_2(\lambda)W_3(\mu)}{W_1(\lambda) - W_1(\lambda-\mu)W_1(\mu)},$$

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$$J_{12} = \frac{W_1(\lambda-\mu)W_2(\lambda)W_1(\mu) - W_2(\lambda-\mu)W_1(\lambda)W_2(\mu)}{W_3(\lambda) - W_3(\lambda-\mu)W_3(\mu)},$$

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Sklyanin Algebra

Actually, the quantities are given by J_{12}, J_{23} and J_{31}

$$J_{12} = \frac{W_1^2(\lambda) - W_2^2(\lambda)}{W_3^2(\lambda) - 1} = (1-\kappa^2)\,\frac{\mathrm{sn}^2(\eta,\kappa)}{\mathrm{cn}^2(\eta,\kappa)\mathrm{dn}^2(\eta,\kappa)},$$

$$J_{23} = \frac{W_2^2(\lambda) - W_3^2(\lambda)}{W_1^2(\lambda) - 1} = \kappa^2\,\frac{\mathrm{sn}^2(\eta,\kappa)\mathrm{cn}^2(\eta,\kappa)}{\mathrm{dn}^2(\eta,\kappa)},$$

$$J_{31} = \frac{W_3^2(\lambda) - W_1^2(\lambda)}{W_2^2(\lambda) - 1} = -\frac{\mathrm{sn}^2(\eta,\kappa)\mathrm{dn}^2(\eta,\kappa)}{\mathrm{cn}^2(\eta,\kappa)}.$$

Sklyanin Algebra

A straightforward calculation shows that

$$J_{12} + J_{23} + J_{31} + J_{12}J_{23}J_{31} = 0.$$

Therefore

$$J_{\alpha\beta} = -\frac{J_\alpha - J_\beta}{J_\gamma},$$

with

$$J_1 : J_2 : J_3 = \frac{\mathrm{cn}(2\eta,\kappa)}{\mathrm{cn}^2(\eta,\kappa)} : \frac{\mathrm{dn}(2\eta,\kappa)}{\mathrm{dn}^2(\eta,\kappa)} : 1.$$

Sklyanin Algebra

Evidently the two dimensional representation of the Sklyanin algebra is given by

$$S^0 = \mathbb{1} \quad \text{and} \quad S^\alpha = \sigma^\alpha.$$

The other irreducible representations of the Sklyanin algebra are constructed by the so-called fusion procedure. In particular, in the three dimensional representation the generators of the Sklyanin algebra are represented by the following set of matrices

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$$S^0 = \begin{pmatrix} J_3 & 0 & J_1 - J_2 \\ 0 & J_1 + J_2 - J_3 & 0 \\ J_1 - J_2 & 0 & J_3 \end{pmatrix},$$

$$S^1 = \sqrt{2J_2J_3} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$S^2 = \sqrt{2J_3J_1} \begin{pmatrix} 0 & -\imath & 0 \\ \imath & 0 & -\imath \\ 0 & \imath & 0 \end{pmatrix},$$

$$S^3 = 2\sqrt{J_1J_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

XYZ Lax Operator

The commutation relations of the generators of the Sklyanin algebra guarantee the RLL-relations for the XYZ Lax operator

$$R_{12}(\lambda-\mu)\mathbb{L}_{1q}(\lambda)\mathbb{L}_{2q}(\mu)=\mathbb{L}_{2q}(\mu)\mathbb{L}_{1q}(\lambda)R_{12}(\lambda-\mu).$$

The XYZ Lax operator satisfies some other important relation, but here we will emphasise the central element of the RLL-relations

$$\mathbb{D}\left[\mathbb{L}(\lambda)\right]=\mathrm{tr}_{00'}P^{-}_{00'}\mathbb{L}_{0q}(\lambda-\eta)\mathbb{L}_{0'q}(\lambda+\eta),$$

where

$$P^{-}_{00'}=\frac{\mathbb{1}-\mathcal{P}_{00'}}{2}=\frac{1}{4}R_{00'}(-2\eta).$$

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Casimirs of the Sklyanin algebra

The central element $\mathbb{D}\left[\mathbb{L}(\lambda)\right]$ can be expressed in terms of the Casimir elements of the Sklyanin algebra

$$\mathbb{D}\left[\mathbb{L}(\lambda)\right] = C_0 - \frac{1 + W_3(\lambda-\eta)W_3(\lambda+\eta)}{J_3}\, C_2,$$

were the quadratic Casimir elements are give by

$$C_0 = (S^0)^2 + \sum_{\alpha=1}^{3} (S^\alpha)^2,$$

$$C_2 = \sum_{\alpha=1}^{3} J_\alpha \, (S^\alpha)^2.$$

Monodromy Matrix

The so-called monodromy matrix

$$T(\lambda) = \mathbb{L}_{0N}(\lambda - \alpha_N) \cdots \mathbb{L}_{01}(\lambda - \alpha_1)$$

is used to describe the system. Notice that $T(\lambda)$ is a two-by-two matrix in the auxiliary space $V_0 = \mathbb{C}^2$, whose entries are operators acting in $\mathcal{H}$

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}.$$

RTT-relations

From RLL-relations it follows that the monodromy matrix satisfies the RTT-relations

$$R_{00'}(\lambda-\mu)T_0(\lambda)T_{0'}(\mu)=T_{0'}(\mu)T_0(\lambda)R_{00'}(\lambda-\mu).$$

The RTT-relations define the commutation relations for the entries of the monodromy matrix.

In the periodic case the modified Algebraic Bethe Ansatz (Takhtajan and Faddeev '79, Takebe '92) yields the spectrum of the spin-s XYZ Heisenberg Hamiltonian

$$H=-\frac{1}{2}\sum_{m=1}^{N}\left(J_1S_m^1S_{m+1}^1+J_2S_m^2S_{m+1}^2+J_3S_m^3S_{m+1}^3\right).$$

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Reflection Equation

A way to introduce non-periodic boundary conditions which are compatible with the integrability of the bulk system, was developed by Sklyanin '88.

The compatibility condition between the bulk and the boundary of the system takes the form of the so-called reflection equation. It is written in the following form for the left reflection matrix acting on the space $V_1 = \mathbb{C}^2$ at the first site, $K^-(u) \in \mathrm{End}(\mathbb{C}^2)$

$$R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v) = K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v).$$

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Reflection Equation

The general solution of the reflection equation above can be written as follows (Vega and Gonzalez '94, Inami and Konno '94, Komori and Hikami '97)

$$K^{-}(u) = \begin{pmatrix} \mathrm{sn}(u+a) - d\,\mathrm{sn}(u-a) & b\,\mathrm{sn}(2u)\,\dfrac{c(1-\tau\,\mathrm{sn}^2(u)) + 1 + \tau\,\mathrm{sn}^2(u)}{1-\tau^2\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(a)} \\ b\,\mathrm{sn}(2u)\,\dfrac{c(1-\tau\,\mathrm{sn}^2(u)) - 1 - \tau\,\mathrm{sn}^2(u)}{1-\tau^2\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(a)} & -\mathrm{sn}(u-a) + d\,\mathrm{sn}(u+a) \end{pmatrix},$$

here a, b, c, d are arbitrary constants.

Dual Reflection Equation

Due to the properties of the Yang R-matrix the dual reflection equation can be presented in the following form

$$R_{12}(v-u)K_1^+(u)R_{21}(-u-v-2\omega)K_2^+(v) =$$

$$= K_2^+(v)R_{12}(-u-v-2\omega)K_1^+(u)R_{21}(v-u).$$

One can then verify that the mapping

$$K^+(u) = K^-(-u-\omega)$$

is a bijection between solutions of the reflection equation and the dual reflection equation. After substitution of into the dual reflection equation one gets the reflection equation with shifted arguments.

Monodromy Matrix $\mathcal{T}(\lambda)$

We use the Sklyanin approach to integrable spin chains with non-periodic boundary conditions. The Sklyanin monodromy matrix $\mathcal{T}(\lambda)$ is

$$\mathcal{T}_0(\lambda) = T_0(\lambda) K_0^-(\lambda) \widetilde{T}_0(\lambda).$$

The monodromy matrix $\widetilde{T}_0(\lambda)$ is such that its RTT-relations can be recast as follows

$$\widetilde{T}_{0'}(\mu) R_{00'}(\lambda+\mu) T_0(\lambda) = T_0(\lambda) R_{00'}(\lambda+\mu) \widetilde{T}_{0'}(\mu),$$
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Reflection Equation Algebra

Then, by construction, the exchange relations of the monodromy matrix $\mathcal{T}(\lambda)$ are

$$R_{00'}(\lambda-\mu)\mathcal{T}_0(\lambda)R_{0'0}(\lambda+\mu)\mathcal{T}_{0'}(\mu)=\mathcal{T}_{0'}(\mu)R_{00'}(\lambda+\mu)\mathcal{T}_0(\lambda)R_{0'0}(\lambda-\mu).$$

Sklyanin determinant

The Reflection Equation Algebra admits a central element, the so-called Sklyanin determinant,

$$\Delta\left[\mathcal{T}(\lambda)\right] = \mathrm{tr}_{00'} P^{-}_{00'} \mathcal{T}_0(\lambda - \eta/2) R_{00'}(2\lambda) \mathcal{T}_{0'}(\lambda + \eta/2).$$

Transfer Matrix

The open chain transfer matrix is given by the trace of the monodromy $\mathcal{T}(\lambda)$ over the auxiliary space V_0 with an extra reflection matrix $K^+(\lambda)$,

$$t(\lambda) = \mathrm{tr}_0 \left(K^+(\lambda) \mathcal{T}(\lambda) \right) .$$

The reflection matrix $K^+(\lambda)$ is the corresponding solution of the dual reflection equation.

The commutativity of the transfer matrix for different values of the spectral parameter

$$[t(\lambda), t(\mu)] = 0,$$

is guaranteed by the dual reflection equation and the exchange relations of the monodromy matrix $\mathcal{T}(\lambda)$.

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The reflection matrix $K^+(\lambda)$ is the corresponding solution of the dual reflection equation.

The commutativity of the transfer matrix for different values of the spectral parameter

$$[t(\lambda), t(\mu)] = 0,$$

is guaranteed by the dual reflection equation and the exchange relations of the monodromy matrix $\mathcal{T}(\lambda)$.

Transfer Matrix

In the spin-$\frac{1}{2}$ case, in the homogeneous limit, this transfer matrix yields the Hamiltonian with the boundary tems

$$H = \sum_{m=1}^{N-1} H_{m,m+1} + \left(A_- \sigma_1^z + B_- \sigma_1^+ + C_- \sigma_1^-\right) + \left(A_+ \sigma_N^z + B_+ \sigma_N^+ + C_+ \sigma_N^-\right) .$$

The spectrum of the transfer matrix was obtained by S. Faldella and G. Niccoli 2014 J. Phys. A: Math. Theor. 47 115202 by the separation of variables method.

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Outline

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Quasi-classical Limit

We observe that the initial R-matrix admits the following expansion

$$R(\lambda, \eta) = \mathbb{1} + 2\eta r(\lambda) + \mathcal{O}(\eta^2),$$

where the classical r-matrix is given by (Sklyanin and Takebe '96)

$$r(\lambda) = \sum_{\alpha=1}^{3} w_\alpha(\lambda)\ \sigma^\alpha \otimes \sigma^\alpha,$$

where

$$w_1(\lambda) = \frac{\mathrm{cn}(\lambda, \kappa)}{\mathrm{sn}(\lambda, \kappa)}, \quad w_2(\lambda) = \frac{\mathrm{dn}(\lambda, \kappa)}{\mathrm{sn}(\lambda, \kappa)}, \quad w_3(\lambda) = \frac{1}{\mathrm{sn}(\lambda, \kappa)}.$$

Classical r-matrix

Evidently, this classical r-matrix has the parity invariance property

$$r_{21}(\lambda) = r_{12}(\lambda),$$

and, due to the fact that $w_\alpha(\lambda)$ are odd function of λ, it also has the unitarity property

$$r_{21}(-\lambda) = -r_{12}(\lambda).$$

Notice that in this case the classical Yang-Baxter equation

$$[r_{13}(\lambda), r_{23}(\mu)] + [r_{12}(\lambda - \mu), r_{13}(\lambda) + r_{23}(\mu)] = 0,$$

reduces to the following three identities

$$w_1(\lambda)\ w_2(\mu) = -w_2(\lambda - \mu)\ w_3(\lambda) + w_1(\lambda - \mu)\ w_3(\mu),$$
$$w_3(\lambda)\ w_1(\mu) = -w_1(\lambda - \mu)\ w_2(\lambda) + w_3(\lambda - \mu)\ w_2(\mu),$$
$$w_2(\lambda)\ w_3(\mu) = -w_3(\lambda - \mu)\ w_1(\lambda) + w_2(\lambda - \mu)\ w_1(\mu),$$

which are consequences of the definition of the functions $w_\alpha(\lambda)$ and the addition theorems of the Jacobi elliptic functions.

Gaudin Lax Operator

The Lax operator of the chain admits the following expansion

$$\mathbb{L}_{0q}(\lambda, \eta) = \mathbb{1} + 2\eta \; \ell_{0q}(\lambda) + \mathcal{O}(\eta^2),$$

where

$$\ell_{0q}(\lambda) = \sum_{\alpha=1}^{3} w_\alpha(\lambda) \; \sigma_0^\alpha \otimes S^\alpha .$$

Therefore the expansion of the monodromy matrix reads

$$T_0(\lambda, \eta) = \mathbb{1} + 2\eta \; L_0(\lambda) + \eta^2 \, T_0^{(2)}(\lambda) + \mathcal{O}(\eta^3),$$

where the Gaudin Lax operator is given by

$$L_0(\lambda) = \sum_{m=1}^{N} \ell_{0n}(\lambda - \alpha_m).$$

Gaudin Lax Operator

The RTT-relations imply the so-called Sklyanin linear bracket for the Gaudin Lax operator

$$[L_0(\lambda), L_{0'}(\mu)] = [r_{00'}(\lambda - \mu), L_0(\lambda) + L_{0'}(\mu)] \, ,$$

with the above classical r-matrix.

Quasi-classical Limit

It can be shown that the transfer matrix of the chain and the quantum determinant of the monodromy matrix admit the following expansions

$$t(\lambda,\eta) = 1 + \eta^2 \; \mathrm{tr}_0 T_0^{(2)}(\lambda) + \mathcal{O}(\eta^3),$$

$$\mathbb{D}\left[T_0(\lambda,\eta)\right] = 1 + \eta^2 \; \left(\mathrm{tr}_0 T_0^{(2)}(\lambda) + 4 \; \mathrm{tr}_0 L_0^2(\lambda)\right) + \mathcal{O}(\eta^3).$$

Thus the generating function $\tau(\lambda)$ of the Gaudin Hamiltonians in the elliptic case can be obtain as a difference

$$\mathbb{D}\left[T_0(\lambda,\eta)\right] - t(\lambda,\eta) = 4\eta \; \mathrm{tr}_0 L_0^2(\lambda) + \mathcal{O}(\eta^3),$$

with, as expected,

$$\tau(\lambda) = \mathrm{tr}_0 L_0^2(\lambda).$$

Evidently, $\tau(\lambda)$ commute for different values of the spectral parameter,

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Gaudin Hamiltonians

As in the rational and the trigonometric case, the expansion into partial fractions yields the corresponding Gaudin Hamiltonians

$$\tau(\lambda) = \sum_{m=1}^{N} \wp(\lambda - \alpha_m)\ s_m(s_m + 1) + \sum_{m=1}^{N} \zeta(\lambda - \alpha_m)\ H_m + H_0,$$

where

$$H_m = 2 \sum_{m \neq n} \sum_{\alpha=1}^{3} w_\alpha(\alpha_n - \alpha_m)\ S_n^\alpha S_m^\beta.$$

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Non-Unitary Classical r-matrix

To define the Gaudin model with boundary terms we consider the following non-unitary classical r-matrix

$$r^K_{00'}(\lambda,\mu) = r_{00'}(\lambda-\mu) - K_{0'}(\nu) r_{00'}(\lambda+\mu) K^{-1}_{0'}(\mu),$$

where

$$K_0(\lambda) \equiv K^-_0(\lambda).$$

It is straightforward to check that this r-matrix satisfies the classical Yang-Baxter equation

$$[r^K_{32}(\lambda_3,\lambda_2), r^K_{13}(\lambda_1,\lambda_3)] + [r^K_{12}(\lambda_1,\lambda_2), r^K_{13}(\lambda_1,\lambda_3) + r^K_{23}(\lambda_2,\lambda_3)] = 0.$$

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Lax Operator in the Boundary Case

The corresponding Lax operator is given by

$$\mathcal{L}_0(\lambda) = \sum_{m=1}^{N} \left(\ell_{0m}(\lambda - \alpha_m) + K_0(\lambda) \ell_{0m}(\lambda + \alpha_m) K_0^{-1}(\lambda) \right) .$$

Evidently, it satisfies the following linear bracket relations

$$\left[\mathcal{L}_0(\lambda), \mathcal{L}_{0'}(\mu) \right] = \left[r^K_{00'}(\lambda, \mu), \mathcal{L}_0(\lambda) \right] - \left[r^K_{0'0}(\mu, \lambda), \mathcal{L}_{0'}(\mu) \right] .$$

By definition this linear bracket is obviously anti-symmetric. It obeys the Jacobi identity because the r-matrix satisfies the classical Yang-Baxter equation.

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Generating Gunction

The generating function $\tau(\lambda)$ of the Gaudin Hamiltonians with boundary terms is given by

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Work in Progress

- Study of the algebra generated by the linear bracket.
- The main aim is the spectrum of the generating function of the Gaudin Hamiltonians with the boundary terms by the suitable modified Algebraic Bethe Ansatz.
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