Piecewise flat metrics and quantum gravity

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1 Regge calculus

- Regge discretization of GR

$$M \to T(M)\,, \quad g_{\mu\nu} \to \{L_\epsilon > 0 \,|\, \epsilon \in T(M)\}\,,$$

such that $g_{\mu\nu}$ is eucledean and flat in each 4-symplex $\sigma \in T(M)$.

- What is $g_{\mu\nu}(\sigma)$? Use Cayley-Menger metric

$$G_{\mu\nu}(\sigma) = L_{0\mu}^2 + L_{0\nu}^2 - L_{\mu\nu}^2\,,$$

where $\sigma = \langle 01234 \rangle$ and $\mu, \nu = 1, 2, 3, 4$. Then

$$g_{\mu\nu}(\sigma) = \frac{G_{\mu\nu}(\sigma)}{(\det G(\sigma))^{1/4}}\,.$$

- Restrictions

$$\det G(\sigma) > 0\,, \quad \det G(\tau) > 0\,, \quad \det G(\Delta) > 0\,(\textit{triangular inequalities})\,,$$

so that one can define the volumes of symplexes

$$\det G(\sigma_n) = 2^n (n!)^2 V^2(\sigma_n)\,, \quad n \in \{2, 3, 4\}\,.$$

- Note that for an arbitrary assignment of L_ϵ the volumes can be positive, zero or imaginary.

- Einstein-Hilbert action

$$\int_M \sqrt{g}\, R\, d^4x \to S_R(L) = \sum_\Delta A_\Delta \delta_\Delta\,,$$

where

$$\delta_\Delta = 2\pi - \sum_\sigma \theta_\Delta^{(\sigma)}$$

and

$$\sin \theta_\Delta^{(\sigma)} = \frac{4}{3} \frac{A_\Delta V_\sigma}{V_\tau V_{\tau'}}\,.$$

- Regge path integral

$$Z = \int_D \prod_{\epsilon=1}^{N_1} dL_\epsilon \, \mu(L) \, e^{-S_R(L)/l_P^2} \, ,$$

where $D \subset (\mathbf{R}_+)^{N_1}$ and

$$\mu(L) = \prod_{\epsilon=1}^{N_1} (L_\epsilon)^\alpha \ , \quad \alpha = const. \quad .$$

- Problem with Z : $S_R(L)$ is not bounded nor has a fixed sign. Use

$$Z_C = \int_D \prod_{\epsilon=1}^{N_1} dL_\epsilon \, \mu(L) \, e^{iS_R(L)/l_P^2} \, ,$$

but there is a problem of how to do Wick's rotation.

2 Minkowski PL metric

- L_ϵ^2 can be negative, so that $L_\epsilon \in \mathbf{R}_+$ or $L_\epsilon \in i\,\mathbf{R}$ $\Rightarrow$ we have to indicate in $T(M)$ which edges are space-like (S) and which edges are time-like (T). We do not use the light-like edges ($L_\epsilon^2 = 0$).

- Restrictions

$$\det G(\sigma) < 0 \quad ,$$
$$\det G(\tau) > 0 \, for \, \tau \in SSS \, or \, \det G(\tau) < 0 \, for \, \tau \in SST \quad ,$$
$$\det G(\Delta) > 0 \, for \, \Delta \in SS \, or \, \det G(\Delta) < 0 \, for \, \Delta \in ST \quad .$$

- Volumes

$$(V_n)^2 = \frac{|\det G_n|}{2^n (n!)^2} > 0 \, , \quad n = 2,3,4 \, ,$$

so that $V_n > 0$.

- Dihedral angles: let

$$(v_n)^2 = \frac{\det G_n}{2^n (n!)^2} \, , \quad n = 2,3,4 \, ,$$

so that $v_n = V_n$ or $v_n = i\,V_n$, while $v_\epsilon = L_\epsilon$ or $v_\epsilon = iL_\epsilon$ where $L_\epsilon > 0$. Then

$$\sin \alpha_\pi^{(\Delta)} = \frac{2\, v_\Delta}{v_\epsilon \, v_{\epsilon'}} \, , \quad \sin \phi_\epsilon^{(\tau)} = \frac{3}{2} \frac{v_\epsilon \, v_\tau}{v_\Delta \, v_{\Delta'}} \, , \quad \sin \theta_\Delta^{(\sigma)} = \frac{4}{3} \frac{v_\Delta \, v_\sigma}{v_\tau \, v_{\tau'}} \quad .$$

- Angles in ST planes: use

$$\cos\alpha = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\,||\vec{v}||}\,,\quad \sin\alpha = \sqrt{1-\cos^2\alpha}\,,\quad \alpha\in\mathbf{C}\,.$$

  Hence for $a\in\mathbf{R}$

  i) $\vec{u}=(1,0)$, $\vec{v}=(\cosh a,\sinh a)$,

$$\cos\alpha = \cosh a\,,\quad \sin\alpha = i\sinh a \;\Rightarrow\; \alpha = i\,a\,.$$

  ii) $\vec{u}=(1,0)$, $\vec{v}=(\sinh a,\cosh a)$,

$$\cos\alpha = \sinh a\,,\quad \sin\alpha = \cosh a \;\Rightarrow\; \alpha = \frac{\pi}{2} + i\,a\,.$$

  iii) $\vec{u}=(0,1)$, $\vec{v}=(\sinh a,\cosh a)$,

$$\cos\alpha = \cosh a\,,\quad \sin\alpha = i\sinh a \;\Rightarrow\; \alpha = i\,a\,.$$

- Dihedral angle θ_Δ

  i) $\sigma=(4,1)\;\Rightarrow\;\sin\theta=\sin a$ for $\Delta\in ST$, $\sin\theta=\cosh a$ for $\Delta\in SS$

  ii) $\sigma=(3,2)\;\Rightarrow\;\sin\theta=i\sinh a$ for $\Delta\in SS$, $\sin\theta=\sin a$ for $\Delta\in ST$

- The deficit angle

$$\delta_\Delta = 2\pi - \sum_\sigma \theta_\Delta^{(\sigma)} \in \mathbf{R}\,for\,\Delta\in ST\,,$$

$$\delta_\Delta = 2\pi - \sum_\sigma \theta_\Delta^{(\sigma)} \in \frac{\pi}{2}\mathbf{Z} + i\mathbf{R}\,for\,\Delta\in SS\,.$$

- Regge action

$$S_R = \sum_{\Delta\in SS} A_\Delta\,\frac{1}{i}\,\delta_\Delta + \sum_{\Delta\in ST} A_\Delta\,\delta_\Delta \in \mathbf{R}\,? \tag{1}$$

  R. Loll et al verified (1) for special triangulations, but one can simply take

$$Re\left(\sum_{\Delta\in SS} A_\Delta\,\frac{1}{i}\,\delta_\Delta\right) = \sum_{\Delta\in SS} A_\Delta\,a_\Delta\,,$$

  which is consistent with the definition of the Lorentzian angles introduced by J.W. Barrett et al.

3 Path-integral quantization

- Take $M = \Sigma \times [0, n]$ and a time-ordered (causal) triangulation

$$T(M) = \cup_{k=0}^{n} T_k(\Sigma) \cup T(B)\,,$$

such that $v_\epsilon = L_\epsilon$ for $\epsilon \in T_k(\Sigma)$ and $v_\epsilon = iL_\epsilon$ for $\epsilon \in T(B)$. The path integral is given by

$$Z = \int_D \prod_{\epsilon=1}^{N_1} dL_\epsilon \, \mu(L) \, e^{iS_R(L)/l_P^2}\,, \tag{2}$$

where $D \subset (\mathbf{R}_+)^{N_1}$ and the measure μ can be any function which makes Z convergent.

- However, if we want that the corresponding effective action $\Gamma(L)$ becomes $S_R(L)$ in the classical limit ($L_\epsilon \gg l_P$), one can show that

$$\ln \mu(\lambda\, L) \approx O(\lambda^a)\,, \quad a \geq 2\,,$$

for $\lambda \to +\infty$, see [1].

- A simple choice for μ, consistent with the diffeomorphism invarince of the semiclassical action is

$$\mu(L) = \exp\left(-V_4(L)/L_0^4\right)\,,$$

where V_4 is the volume of $T(M)$ and L_0 is a new parameter in the theory, which can be fixed by requiring that the effective cosmological constant coincides with the observed value, see [2].

- The path integral (2) is a function of the initial edge lengths l_ϵ on $T_0(\Sigma)$ and the final edge lengths l'_ϵ on $T_n(\Sigma)$. This is known as the propagator, $G(l, l')$, since it represents the propagator for the third-quantized theory, $\Psi(l) \to \Phi[\Psi(l)]$.

4 The wavefunction of the Universe

- In canonical QG there is a wavefunction $\Psi(h)$, which satisfies the WdW equation

$$\hat{W}(\hat{p}_h, \hat{h})\Psi(h) = 0\,,$$

where h is a metric on Σ and

$$ds^2 = -(N^2 - n^i n_i)dt^2 + 2n_i\, dt dx^i + h_{ij}\, dx^i dx^j\,,$$

gives the spacetime metric.

- What is $\Psi(h)$ in the PI quantization? Hartle and Hawking:

$$\Psi(h) = Z_E(h)\,, \tag{3}$$

where M has the topology of a cup ($\partial M = \Sigma$) and the metrics on M are euclidian.

- $\Psi(h)$ can be calculated for the minisuperspace models, where the metric has a finite number of DOF, for example, the FLRW metric

$$ds^2 = -N^2(t)\, dt^2 + a^2(t)\, (dx^2 + dy^2 + dz^2)\,. \tag{4}$$

Consequently

$$\Psi(a) = \int \mathcal{D}N \int \mathcal{D}a \exp\left(-\int_I dt\, L_E(a, \dot{a}, N)/l_P^2\right)\,. \tag{5}$$

- In general case the path integral (5) can be calculated only approximately by using the stationary phase aproximation. In the case of the FRLW metric, one can obtain a solution of the WdW equation if a special choice of the contour of integration for N is made.

- The HH wavefunction can be calculated in the PLQG formulation, and the advantage is that there are no ambigous or complex domains of integration.

- Example: $M = S^4$, $\Sigma = S^3$, $T(M) = T(S^3) \cup T(cone)$, $L_\epsilon = l > 0\,, \epsilon \in T(S^3)$, $L_\epsilon = s > 0\,, \epsilon \in T(cone)$, so that

$$\Psi(l) = \int_0^\infty ds\, \mu(l,s)\, \exp\left(iS_R(l,s)/l_P^2\right)\,.$$

- For $T(S^3)$ a pentagon (5 tetrahedrons)

$$S_R(l,s) = \frac{5\sqrt{3}}{2}\, l^2\, \delta_1(l,s) + \frac{5}{2}\, l\sqrt{s^2 - \frac{l^2}{4}}\, \delta_2(l,s)\,,$$

where

$$\delta_1 = 2\pi - 3\alpha - \gamma\,, \quad \delta_2 = 2\pi - 3\beta\,,$$

$$\sin\alpha = \frac{\sqrt{s^2 - \frac{3l^2}{8}}}{\sqrt{s^2 - \frac{l^2}{3}}}\,, \quad \sin\beta = \frac{2\sqrt{2}\sqrt{s^2 - \frac{3l^2}{8}}\sqrt{s^2 - \frac{l^2}{4}}}{3(s^2 - \frac{l^2}{3})}\,, \quad \sin\gamma = \frac{\sqrt{3}}{4}$$

and

$$\mu(l,s) = \exp\left(-\frac{l^3}{L_0^4}\sqrt{s^2 - \frac{3l^2}{8}}\right)\,.$$

- Note that

$$l = a l_0\,, \quad s = N t_0 \quad .$$

- Is there a WdW operator $\hat{W}$ such that $\hat{W}\Psi(a) = 0$?

$$\Leftrightarrow \hat{W}\Psi = \alpha\frac{1}{a}\frac{d^2\Psi}{da^2} + \beta\frac{d}{da}\left(\frac{1}{a}\frac{d\Psi}{da}\right) + \gamma\frac{d^2}{da^2}\left(\frac{\Psi}{a}\right) + a^3\Psi = 0\,,$$

for some $\alpha, \beta, \gamma \in \mathbf{R}$.

- This is not necessarilly true in PLQG, becase WdW equation corresponds to a smooth manifold M, while we have a PL manifold $T(M)$. However, when $N_1 \to \infty$ (smooth limit)

$$\hat{W}_{T(M)} \to \hat{W}_M\,.$$

- Bosonic matter can be coupled via the PL metric $g_{\mu\nu}(\sigma)$. In the case of a scalar field ϕ

$$S_m = \sum_\sigma V_\sigma\,\mathcal{L}_\sigma\,,$$

where

$$\mathcal{L}_\sigma = -\frac{1}{2}g^{\mu\nu}(\sigma)\,D_\mu\phi\,D_\nu\phi \;-\; U(\phi_0)\,.$$

$g^{\mu\nu}(\sigma)$ is the inverse metric and

$$D_\mu\phi = \frac{\phi_\mu - \phi_0}{L_{0\mu}}\,,$$

where $\phi_\mu = \phi(\pi_\mu)$ and $\phi_0 = \phi(\pi_0)$.

- The HH wavefuction for $T(M) = T(S^3) \cup T(cone)$

$$\Psi(l,f) = \int_0^\infty ds \int_{-\infty}^\infty d\varphi \, \mu(l,s) \exp\left(\frac{i}{l_P^2}\left[S_R(l,s) + S_m(l,s,\varphi,f)\right]\right) ,$$

where $\phi(\pi) = f$ for $\pi \in T(S^3)$ and $\phi(\pi) = \varphi$ for $\pi \in T'(cone)$.

- Fermionic matter can be coupled via the PL tetrads $e_\mu^a(\sigma)$

$$\eta_{ab}\, e_\mu^a(\sigma)\, e_\nu^b(\sigma) = g_{\mu\nu}(\sigma)$$

and PL spin connections $\omega_\mu^{ab}(\sigma)$.

- An alternative way to construct the WFU: the propagator

$$G(h,h') = Z(h,h')$$

is the kernel of the WdW equation

$$\hat{W}(\hat{p}_h, \hat{h})\, G(h,h') = \delta(h - h') \,.$$

Hence

$$\Psi(h) = \int \mathcal{D}h' \, G(h,h')\, \Psi_0(h') \,,$$

where $\Psi_0(h)$ is the initial WFU.

- Note that $\Psi_0(h)$ has to satisfy $\hat{W}\Psi_0 = 0$.

- In the PLQG case $G(h,h') \to G(l,l') = Z(l,l')$.

- PLQG example:

$$M = S^3 \times [0,1] \,, \quad T(M) = T_0(S^3) \cup T(B) \cup T_1(S^3)$$

where $T_0 = T_1$, all the edges in T_0 and T_1 are spacelike, while the edges in $T(B)$ can be all timelike or all spacelike.

- Toy model: $T_0 = T_1 = \sigma_5$, $A_j \in T_0$ and $B_j \in T_1$, $j = 1,2,...,5$ and

$$||A_j\vec{A}_k|| = l' \,, \quad ||B_j\vec{B}_k|| = l \,, \quad ||A_j\vec{B}_j||^2 = \frac{2}{5}(l - l')^2 - t^2 \,, \quad t \geq 0 \,.$$

Hence

$$G(l,l') = \int_0^\infty dt\, \mu(l,l',t) \, \exp\left(iS_R(l,l',t)/l_P^2\right) \,.$$

5 Conclusions

- Replacing $D(T(M), L)$ with $\mathbf{R}_+^{N_1}$?
- Smooth-limit approximations of MS models?
- HH vs CQG wavefunction.

References

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