Wormhole Modeling in General Relativity

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Outline

1. Wormholes act as tunnels from one region of spacetime to another, possibly through which observers may freely traverse.
2. Although we have a vague image of wormhole, there is no universal definition which can work for general situations.
3. The idea is essential in science fictions as a way for rapid interstellar travel, warp drives, and time machines. However, wormhole is also a theoretical research topic with long history.

1. **Einsteins General Theory of Relativity (GTR, 1915):** Einstein's theory is that space and time can warp into each other. In particular, the curvature of spacetime is directly related to the energy and momentum of whatever matter and radiation are present- physics is in the fabric of space-time.
2. **Einstein-Rosen Bridge (1935):** They constructed an elementary particle model represented by a "bridge" connecting two identical sheets. This mathematical representation of physical space being connected by a wormhole type solution was denoted an "Einstein-Rosen bridge".
3. **John Wheeler (ca. 1957,1962):** Wheeler considered wormholes, such as Reissner-Nordström or Kerr wormholes, as objects of the quantum foam connecting different regions of spacetime and operating at the Planck scale. He first introduced the the word "wormhole".

"Morris-Thorne framework" Am. J. Phys. 56, 395 (1988).

We should first begin by discussing the criteria for construction of traversable wormholes:

1. Metric should be both spherically symmetric and static. This is just to keep everything simple.
2. Solution must everywhere obey the Einstein field equations. This assumes correctness of GTR.
3. Solution must have a throat that connects two asymptotically flat regions of spacetime.
4. No horizon, since a horizon will prevent two-way travel through the wormhole.

According to Morris and Thorne this is called **"basic wormhole criteria"**.

Criteria for Construction Wormhole Cont.

5. Tidal gravitational forces experienced by a traveler must be negligible.
6. Traveler must be able to cross through the wormhole in a finite and reasonably small proper time.
7. Physically reasonable stress-energy tensor generated by the matter and fields.
8. Solution must be stable under small perturbation.
9. Should be possible to assemble the wormhole, i. e. assembly should require both much less than the total mass of the universe and much less than the age of the universe.

This is **usability criteria** of wormhole construction. since it deals with human physiological comfort.

The general static spherically symmetric wormhole solution with usual spherical coordinates (t, r, θ, ϕ), we have the general metric

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (1)$$

1. Φ is known as **"redshift function"**-related to the gravitational redshift.
2. $b(r)$- is the **"shape function"**- it determines the shape of the wormhole.
3. The coordinate r decreases from $+\infty$ to a minimum value r_0, representing the location of the **throat of the wormhole**, where $b(r_0) = r_0$, and then it increases from r_0 to $-\infty$.
4. proper circumference of a circle of fixed r is given by $2\pi r$.

Wormhole Modeling Cont.

An alternative way of expressing the above metric is

$$ds^2 = -e^{2\Phi}dt^2 + dl^2 dr^2 + r^2(l)(d\theta^2 + \sin^2\theta d\phi^2) \qquad (2)$$

where we have set the **proper radial distance** as

$$L(r) = \pm \int_{r_0}^{r} \frac{dr}{\sqrt{1 - \frac{b(r)}{r}}} \qquad (3)$$

- which is required to be finite everywhere.
- $L(r)$ decreases from $+\infty$ in the upper universe, to $L = 0$ at the throat, and then from zero to $-\infty$ in the lower universe.
- For the wormhole to be traversable it must have no horizons, which implies that $g_{tt} = e^{2\Phi} \neq 0$, so that $\Phi(r)$ must be finite everywhere.

The mathematics of embedding and generic static throat

We can use embedding diagrams to represent a wormhole and extract some useful information for the choice of the shape function, $b(r)$ and one may consider an equatorial slice, $\theta = \frac{\pi}{2}$, with a some fixed moment of time $t = constant$, the metric should be

$$ds^2 = \left(1 - \frac{b(r)}{r}\right)^{-1} dr^2 + r^2 d\phi^2 \qquad (4)$$

To visualize this slice, one embeds this metric into three-dimensional Euclidean space, in which the metric can be written in cylindrical coordinates, (r, ϕ, z), as

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2 \qquad (5)$$

Comparing both equations, we have the equation for the embedding surface, given by

The mathematics of embedding and generic static throat

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$$\frac{dz}{dr} = \pm\left(\frac{r}{b(r)} - 1\right)^{-\frac{1}{2}} \tag{6}$$

- The geometry has a minimum radius, $r = b(r) = r_0$, denoted as the throat.
- Far from the throat consider that space is asymptotically flat, $\frac{dz}{dr} \to 0$, as $r \to \infty$.
- To be a solution of a wormhole, one needs to impose that the **throat flares out**. Mathematically, this flaring-out condition entails that the inverse of the embedding function $r(z)$, must satisfy $\frac{d^2r}{dz^2} > 0$ at or near the throat r_0.

Wormhole Modeling Cont.

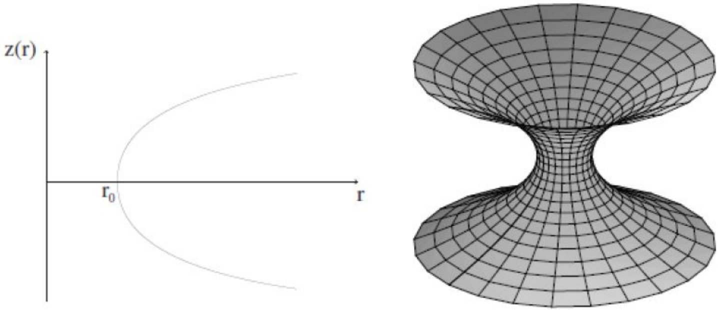
Differentiating $\frac{dr}{dz} = \pm \left(\frac{r}{b(r)} - 1 \right)^{\frac{1}{2}}$ with respect z, we have

$$\frac{d^2 r}{dz^2} = \frac{b - rb^{'}}{2b^2} > 0 \tag{7}$$

At the throat one can verify that the form function satisfies the condition $b^{'}(r_0) < 1$.

- These geometries also allow closed timelike curves, with the respective causality violations. In a closed timelike curve, the worldline of an object through spacetime follows a curious path where it eventually returns to the exact same coordinates in space and time that it was at previously.
- These spacetimes is that they allow "effective" superluminal travel, although, locally, the speed of light is not surpassed.

Wormhole Modeling Cont.



The embedding diagram of a two-dimensional section along the equatorial plane (t $=$ *const*, $\theta = \pi/2$) of a traversable wormhole. For a full visualization of the surface sweep through a 2π rotation around the zaxis.

Einstein Field Equations

The system of equations are obtained as:

$$\rho(r) = \frac{b^{'}}{r^2} - \Lambda, \tag{8}$$

$$\tau(r) = \frac{b}{r^3} - 2\left(1 - \frac{b}{r}\right)\frac{\Phi^{'}}{r} - \Lambda, \tag{9}$$

$$p_t(r) = \left(1 - \frac{b}{r}\right)\left[\Phi^{''} + (\Phi^{'})^2 - \frac{rb^{'} - b}{2r^2(1 - b/r)}\Phi^{'} - \frac{rb^{'} - b}{2r^3(1 - b/r)} + \frac{\Phi^{'}}{r}\right] + \Lambda \tag{10}$$

Here, $\tau(r)$ is the radial tension, with $\tau(r) = -p_r(r)$

The energy conditions for the specific case in which the stress-energy tensor is diagonal i. e.,

$$T^{\mu\nu} = diag(\rho, p_1, p_2, p_3) \tag{11}$$

where ρ is the mass density and the p_i are the three principal pressures.

1. **Null energy condition (NEC):** The NEC asserts that for any null vector k^μ: $T_{\mu\nu}k^\mu k^\nu \geq 0$.
   In the case of a stress-energy tensor of the form Eq. (11) , we have
   $$\rho + p_i \geq 0, \quad \forall i \tag{12}$$

2. **Weak energy condition (WEC)**: The WEC states that for any timelike vector U^{μ}: $T_{\mu\nu}U^{\mu}U^{\nu} \geq 0$.
Thus, the WEC requires that energy density to be positive. In terms of the principal pressures this gives

$$\rho \geq 0, \quad \rho + p_i \geq 0, \forall i \tag{13}$$

Exotic Matter

To gain some insight into the matter threading the wormhole, Morris and Thorne defined the dimensionless function $\xi = \frac{\tau - \rho}{|p|}$. Using field equations one finds

$$\xi = \frac{\tau - \rho}{|p|} = \frac{\frac{b}{r} - b^{'} - 2r(1 - b/r)\Phi^{'}}{|b^{'}|} \tag{14}$$

Considering the finite character of ρ, and therefore of $b^{'}$, and the fact that $(1 - b/r)\Phi^{'} \to 0$ at the throat, we have the following relationship

$$\xi(r_0) = \frac{\rho_0 - \tau_0}{|p_0|} < 0 \tag{15}$$

The restriction $\tau_0 > \rho_0$ is an extremely troublesome condition, as it states that the radial tension at the throat should exceed the energy density. Thus, Morris and Thorne coined matter restricted by this condition **"exotic matter"**

Exotic Matter Cont.

- The wormhole material is everywhere exotic, i.e., $\xi < 0$ everywhere, extending outward from the throat, with ρ, *tau* and p tending to zero as $r \to +\infty$.
- Exotic matter is particularly troublesome for measurements made by observers traversing through the throat with a radial velocity close to the speed of light.
- The energy density measured by these observers is given by $T_{00} = \gamma^2(\rho_0^2 - v^2\tau_0^2)$ with $\gamma^2 = (1 - v^2)^{\frac{-1}{2}}$.
- For sufficiently high velocities, $v \to 1$, the observer will measure a negative energy density, $T_{00} < 0$.
- This feature also holds for any traversable, nonspherical and nonstatic wormhole.

Traversability Conditions

Traveller journeys radially through a wormhole, beginning at rest in a space station in the lower universe, at $l = -l_1$, and ending at rest in a space station in the upper universe at $l = +l_2$.

- Assume that the traveller has a radial velocity $v(r)$, as measured by a static observer positioned at r. One may relate the proper distance travelled dl, radius travelled dr , coordinate time lapse dt, and proper time lapse as measured by the observer $d\tau$, by the following relationships:

$$v = e^{-\Phi}\frac{dl}{dt} = \mp e^{-\Phi}(1 - b/r)^{-1/2}\frac{dr}{dt} \tag{16}$$

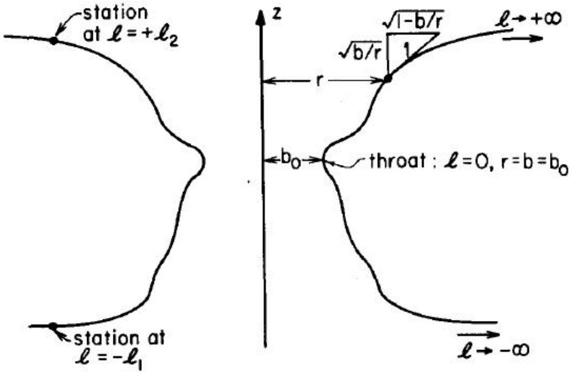
$$v\gamma = \frac{dl}{dt} = \mp(1 - b/r)^{-1/2}\frac{dr}{dt} \tag{17}$$

Traversability Conditions Cont.

It is also important to impose certain conditions at the space stations.

- Consider that space is asymptotically flat at the stations, i. e. $\frac{b}{r} \ll 1$.
- The gravitational redshift of signals sent from the stations to infinity should be small, i. e. $\frac{\triangle\lambda}{\lambda} = e^{-\Phi} - 1 \equiv -\Phi$, so that $|\Phi| \ll 1$. The condition $|\Phi| \ll 1$, imposes that the proper time at the station equals the coordinate time.
- The gravitational acceleration measured at the stations, given by $g = -(1 - b/r)^{\frac{-1}{2}} \Phi' \equiv -\Phi'$ should be less than or equal to the Earths gravitational acceleration, $g \leq g_{\oplus}$, so, that the condition $|\Phi'| \leq g_{\oplus}$.
- The entire journey should be done in a relatively short time as measured both by the traveller and by observers who remain at rest at the stations.
- Acceleration felt by the traveller should not exceed the Earths gravitational acceleration, $g_{\oplus}$.

Traversability conditions



In this work, the shape function $b(r) = \frac{r_0 \log(r+1)}{\log(r_0+1)}$ is considered, and the variable redshift function $\Phi(r)$ is defined as $\Phi(r) = -\frac{1}{r^2}$. Using these shape and redshift functions, the field equations are solved and the energy condition terms are derived which are as follows:

$$\rho = \frac{1}{\omega+1}\left[\frac{4\left(1-\frac{r_0\log(r+1)}{r\log(r_0+1)}\right)}{r^4} - \frac{r_0\log(r+1)}{r^3\log(r_0+1)} + \frac{r_0}{r^2(r+1)\log(r_0+1)} \right. \quad (1$$

$$p_r = \frac{\omega}{\omega+1}\left(\frac{4\left(1-\frac{r_0\log(r+1)}{r\log(r_0+1)}\right)}{r^4} - \frac{r_0\log(r+1)}{r^3\log(r_0+1)} + \frac{r_0}{r^2(r+1)\log(r_0+1)}\right)$$

Solutions Cont.

$$\Lambda = \frac{r_0}{r^2(r+1)\log(r_0+1)} - \frac{1}{\omega+1}\left[\frac{4}{r^4}\left(1-\frac{r_0\log(r+1)}{r\log(r_0+1)}\right) - \frac{r_0\log(r+1)}{r^3\log(r_0+1)} + \frac{r_0}{r^2(r+1)\log(r_0+1)}\right] \tag{20}$$

Solutions Cont.

$$
\begin{aligned}
p_t &= -\frac{4}{r^4}\left(1-\frac{r_0\log(r+1)}{r\log(r_0+1)}\right)+\frac{r_0\log(r+1)}{r^3\log(r_0+1)} \\
&+ \frac{\omega}{\omega+1}\left(\frac{4}{r^4}\left(1-\frac{r_0\log(r+1)}{r\log(r_0+1)}\right)\right. \\
&- \left.\frac{r_0\log(r+1)}{r^3\log(r_0+1)}+\frac{r_0}{r^2(r+1)\log(r_0+1)}\right) \\
&+ \left(1-\frac{r_0\log(r+1)}{r\log(r_0+1)}\right)\left(\frac{4}{r^6}-\frac{1}{r^5\left(1-\frac{r_0\log(r+1)}{r\log(r_0+1)}\right)}\right. \\
&\times \left(\frac{r r_0}{(r+1)\log(r_0+1)}-\frac{r_0\log(r+1)}{\log(r_0+1)}\right)-\frac{4}{r^4}-\frac{1}{2r^3\left(1-\frac{r_0\log(r+1}{r\log(r_0+1}\right.} \\
&- \left.\left.-\frac{r_0\log(r+1)}{\log(r_0+1)}\right)\right)
\end{aligned}
$$

Conclusions

- This work is focused on the investigation of traversable wormholes introduced by Morris and Thorne in the presence of cosmological constant.
- Thorne with his student Morris constructed traversable wormholes with two mouths and one throat.
- They considered static and spherically symmetric wormholes with constant redshift function and, obtained the presence of the **exotic matter** at the throat of the wormholes.
- Eventually, they concluded that the presence of exotic matter at the throat is **necessary** for the construction of traversable wormholes in general relativity, i. e. near the throat of the wormhole the material must hold the radial tension exceed the mass energy density ($\tau_0 > \rho_0 c^2$), which indicates the violation of the null energy condition near the throat of the wormholes.

Conclusions Cont.

- However, in this work we tried to construct a traversable wormhole by avoiding exotic matter. Therefore, in this work, we constructed traversable wormholes in general relativity with cosmological constant by assuming variable redshift and shape functions.
- The main motivation of this work is to minimize the exotic matter near the throat of the wormholes.
- In this work, variable redshift and shape functions are used to construct a traversable wormholes in general relativity with cosmological constant.
- The null, weak, strong and dominated energy conditions are analysed and spherical regions satisfying the null, weak and strong energy conditions with positive cosmological constant are obtained.

Conclusions Cont.

- It is found that wormholes filled with non exotic matter satisfying NEC and WEC with positive value of cosmological constant exists for $r \geq 0.1$.
- Hence, this study concludes the exotic matter could be avoided in the construction of traversable wormholes in general relativity by introducing cosmological constant and suitable choice of variable redshift and shape functions.

THANK YOU