T-duality between effective string theories

Lj. Davidović and B. Sazdović

Institute of Physics,
University of Belgrade, Serbia

10th Mathematical Physics Meting
Belgrade, Serbia

Outline

$$\begin{array}{ccc} \kappa \int d\xi^2 \partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu & \overset{T}{\longleftrightarrow} & \frac{\kappa^2}{2} \int d\xi^2 \partial_+ y_\mu \Theta_-^{\mu\nu} \partial_- y_\nu \\ \Big\downarrow \, \Gamma=0 & & \Big\downarrow \, {}^\star\Gamma=0 \\ \frac{\kappa}{2} \int d\xi^2 \partial_+ Q^\mu G^{eff}_{\mu\nu} \partial_- Q^\nu & \overset{T}{\longleftrightarrow} & \frac{\kappa}{2} \int d\xi^2 \partial_+ K_\mu {}^\star G^{\mu\nu}_{eff} \partial_- K_\nu \end{array}$$

Action

Bosonic string action

$$S[x] = \kappa \int_\Sigma d^2\xi \sqrt{-g} \Big[\frac{1}{2} g^{\alpha\beta} G_{\mu\nu}[x] + \frac{\epsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu}[x] \Big] \partial_\alpha x^\mu \partial_\beta x^\nu$$

Action principle $\delta S = 0$ gives equations of motion and boundary conditions

$$\gamma_\mu^{(0)}(x) \delta x^\mu /_{\sigma=\pi} \quad - \quad \gamma_\mu^{(0)}(x) \delta x^\mu /_{\sigma=0} = 0$$

where we define

$$\gamma_\mu^{(0)}(x) \equiv \frac{\delta S}{\delta x'^\mu} = \kappa \Big(2B_{\mu\nu} \dot{x}^\nu - G_{\mu\nu} x'^\nu \Big)$$

Action 2

- Conformal gauge

$$g_{\alpha\beta} = e^F \eta_{\alpha\beta}$$

- Action

$$S = \kappa \int d\xi^2 \partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu$$

- with the background field composition

$$\Pi_{\pm\mu\nu}(x) = B_{\mu\nu}(x) \pm \frac{1}{2} G_{\mu\nu}(x)$$

- and the light-cone coordinates

$$\xi^\pm = \frac{1}{2}(\tau \pm \sigma) \qquad \partial_\pm = \partial_\tau \pm \partial_\sigma$$

Boundary conditions

- Let us choose the Neumann condition for coordinates x^a, $a = 0, 1, \dots, p$ and the Dirichlet condition for coordinates x^i, $i = p+1, \dots, D-1$,

$$N: \quad \gamma_a^{(0)}\Big|_{\partial\Sigma} = 0, \quad \gamma_a^{(0)} \equiv {}_N\gamma_a^0 = \kappa\big(\Pi_{+ab}\partial_- x^b + \Pi_{-ab}\partial_+ x^b\big)$$

$$D: \quad \kappa\dot{x}^i\Big|_{\partial\Sigma} = 0, \quad {}_D\gamma_0^i \equiv \kappa\dot{x}^i$$

- We consider the block diagonal constant metric and Kalb-Ramond field $G_{\mu\nu} = const$, $B_{\mu\nu} = const$

$$G_{\mu\nu} = \begin{pmatrix} G_{ab} & 0 \\ 0 & G_{ij} \end{pmatrix}, \quad B_{\mu\nu} = \begin{pmatrix} B_{ab} & 0 \\ 0 & B_{ij} \end{pmatrix}$$

Buscher T-duality procedure for constant background 1

- T-duality: Strings propagating on completely different spacetime geometries may be physically equivalent
- Buscher procedure:
  - gauging global symmetries $\delta x^\mu = \lambda^\mu$
    $\partial_\alpha x^\mu \to D_\alpha x^\mu = \partial_\alpha x^\mu + v^\mu_\alpha$,
    - v^μ_α gauge field
    - D_α covariant derivative
  - Field strength $F^\mu_{\alpha\beta} = \partial_\alpha v^\mu_\beta - \partial_\beta v^\mu_\alpha$
  - T-dual theory must be Physically equivalent to initial theory
    $F^\mu_{01} \equiv F^\mu = 0$

Buscher T-duality procedure for constant background 2

- Invariant Action

$$S_{inv}(x, y, v) = \kappa \int_{\Sigma} d^2\xi \left[D_+ x^\mu \Pi_{+\mu\nu} D_- x^\nu + \frac{1}{2} y_\mu F^\mu \right]$$

  - y_μ Lagrange multiplier
- Gauge fixing $x^\mu = 0$
- Gauge fixed Action

$$S_{fix}(y, v) = \kappa \int_{\Sigma} d^2\xi \left[v_+^\mu \Pi_{+\mu\nu} v_-^\nu + \frac{1}{2} y_\mu F^\mu \right]$$

Buscher T-duality procedure for constant background 3

- Check
  y_μ: $\partial_\alpha v^\mu_\beta - \partial_\beta v^\mu_\alpha = 0 \Longrightarrow v^\mu_\alpha = \partial_\alpha x^\mu \Longrightarrow S_{fix} \to S(x)$
- Elimination of gauge fields on equations of motion produces T-dual Action

$$^\star S[y] = \frac{\kappa^2}{2} \int d^2\xi \; \partial_+ y_\mu \theta^{\mu\nu}_- \partial_- y_\nu$$

Buscher T-duality procedure for constant background 4

- Dual Action ${}^\star S(y)$ has the same form as initial one, but with different background fields

$$
{}^\star S[y] = \kappa \int d^2\xi\ \partial_+ y_\mu\, {}^\star\Pi_+^{\mu\nu}\, \partial_- y_\nu = \frac{\kappa^2}{2} \int d^2\xi\ \partial_+ y_\mu \theta_-^{\mu\nu} \partial_- y_\nu
$$

where T-dual background fields

$$
{}^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}, \quad {}^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}
$$

$$
G^E_{\mu\nu} \equiv G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}, \qquad \theta^{\mu\nu} \equiv -\frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu}
$$

$$
\Pi_\pm \equiv B_{\mu\nu} \pm \frac{1}{2}G_{\mu\nu}\,, \qquad \theta_\pm^{\mu\nu} \equiv \theta^{\mu\nu} \mp \frac{1}{\kappa}(G_E^{-1})^{\mu\nu}
$$

T-duality transformation of variables for constant background

- T-dual transformations

$$v_\pm^\mu \cong \partial_\pm x^\mu \cong -\kappa \Theta_\pm^{\mu\nu} \partial_\pm y_\nu$$

- together with inverse transformation produces
T-duality transformation of variables

$$\partial_\pm x^\mu \cong -\kappa \theta_\pm^{\mu\nu} \partial_\pm y_\nu \,, \qquad \partial_\pm y_\mu \cong -2\Pi_{\mp\mu\nu} \partial_\pm x^\nu$$

- in other form

$$-\kappa \, \dot{x}^\mu \cong {}^\star\gamma_{(0)}^\mu(y) \,, \quad \gamma_\mu^{(0)}(x) \cong -\kappa \, \dot{y}_\mu$$

T-dual boundary condition

- T-dual boundary conditions

$$^\star\gamma^{(0)\mu}\,\delta y_\mu\Big|_0^\pi = 0$$

where

$$^\star\gamma^{(0)\mu} = \frac{\kappa^2}{2}\left[\Theta_-^{\mu\nu}\partial_- y_\nu + \Theta_+^{\mu\nu}\partial_+ y_\nu\right]$$

- The T-dual theory is equivalent to a initial open string theory with chosen boundary conditions, if the T-dual boundary conditions are fulfilled in a Neumann way for coordinates y_i and in a Dirichlet way for y_a

$$N: \quad ^\star\gamma^{(0)i}\Big|_{\partial\Sigma} = 0, \qquad {}^\star_N\gamma_0^i = \frac{\kappa^2}{2}\left[\Theta_-^{ij}\partial_- y_j + \Theta_+^{ij}\partial_+ y_j\right]$$

$$D: \quad \kappa\dot{y}_a\Big|_{\partial\Sigma} = 0, \qquad {}^\star_D\gamma_a^0 = \kappa\dot{y}_a$$

T-dual boundary condition 2

This is because of the T-duality transformation law

$$-\kappa\dot{x}^\mu \cong {}^\star\gamma^\mu_{(0)}(y), \quad \gamma^{(0)}_\mu(x) \cong -\kappa\dot{y}_\mu$$

and consequently

$$\begin{aligned} {}_D\gamma_0^i &\equiv \kappa\dot{x}^i \cong -{}^\star\gamma^i_{(0)}(y) \equiv -{}^\star_N\gamma_0^i, \\ {}_N\gamma_a^0 &\equiv \gamma_a^{(0)} \cong -\kappa\dot{y}_a = -{}^\star_D\gamma_a^0 \end{aligned}$$

T-dualization changes the type of the boundary conditions

Boundary conditions in canonical form

- We are going to treat boundary conditions as constraints and apply the Dirac consistency procedure
- Canonical form of boundary conditions

$$
\begin{aligned}
{}_N\gamma_a^0 &= \Pi_{+ab}(G^{-1})^{bc} j_{-c} + \Pi_{-ab}(G^{-1})^{bc} j_{+c}, \\
{}_D\gamma_0^i &= \kappa \dot{x}^i = \frac{1}{2}(G^{-1})^{ij}(j_{+j} + j_{-j})
\end{aligned}
$$

can be expressed in terms of currents

$$
j_{\pm\mu} = \pi_\mu + 2\kappa \Pi_{\pm\mu\nu} x'^\nu
$$

Dirac consistency procedure applied to the boundary conditions

- The algebra of currents in a constant background

$$\{j_{\pm\mu}(\sigma), j_{\pm\nu}(\bar\sigma)\} = \pm\, 2\kappa\, G_{\mu\nu}\, \delta'(\sigma - \bar\sigma),$$
$$\{j_{\pm\mu}(\sigma), j_{\mp\nu}(\bar\sigma)\} = 0$$

- Following the Dirac procedure, one can impose consistency to these constraints. The additional constraints are defined for every $n \geq 1$

$${}_N\gamma_a^n = \{H_C, {}_N\gamma_a^{n-1}\}, \quad {}_D\gamma_n^i = \{H_C, {}_D\gamma_{n-1}^i\}$$

with $H_C = \int d\sigma \mathcal{H}_C$ is canonical hamiltonian

σ-dependent constraints

- All these constraints can be gathered into only two constraints

$$\Gamma_a^N(\sigma) = \sum_{n\geq 0} \frac{\sigma^n}{n!} \, {}_N\gamma_a^n \Big|_{\sigma=0}, \quad \Gamma_D^i(\sigma) = \sum_{n\geq 0} \frac{\sigma^n}{n!} \, {}_D\gamma_n^i \Big|_{\sigma=0}$$

- We obtain the explicit form of the sigma dependent constraints

$$\Gamma_a^N(\sigma) = \Pi_{+ab}(G^{-1})^{bc} j_{-c}(\sigma) + \Pi_{-ab}(G^{-1})^{bc} j_{+c}(-\sigma),$$
$$\Gamma_D^i(\sigma) = \frac{1}{2}(G^{-1})^{ij}\Big[j_{+j}(-\sigma) + j_{-j}(\sigma)\Big]$$

- If we demand 2π-periodicity

$$x^\mu(\sigma + 2\pi) = x^\mu(\sigma),$$
$$\pi_\mu(\sigma + 2\pi) = \pi_\mu(\sigma)$$

$\sigma-$dependent constraints for $\sigma = 0$ and $\sigma = \pi$ are equal

- They are of the second class and one can solve them

Independent canonical variables

- Divide canonical variables into even and odd parts

$$x^\mu = q^\mu + \bar{q}^\mu, \quad \pi_\mu = p_\mu + \bar{p}_\mu$$

$$q^\mu = \sum_{n\geq 0} \frac{\sigma^{2n}}{(2n)!} x^{(2n)\mu}\Big|_{\sigma=0}, \quad \bar{q}^\mu = \sum_{n\geq 0} \frac{\sigma^{2n+1}}{(2n+1)!} x^{(2n+1)\mu}\Big|_{\sigma=0}$$

$$p_\mu = \sum_{n\geq 0} \frac{\sigma^{2n}}{(2n)!} \pi_\mu^{(2n)}\Big|_{\sigma=0}, \quad \bar{p}_\mu = \sum_{n\geq 0} \frac{\sigma^{2n+1}}{(2n+1)!} \pi_\mu^{(2n+1)}\Big|_{\sigma=0}$$

Solution of the constraints

- Requiring
$$\Gamma_a^N(\sigma) = 0, \quad \Gamma_D^i(\sigma) = 0$$
one obtains the solution
$$\bar{p}_a = 0, \qquad \bar{q}'^a = -\theta^{ab} p_b,$$
$$q'^i = 0, \qquad p_i = -2\kappa B_{ij} \bar{q}'^j$$

Solution of the constraints 2

- Solving the constraints has reduced the phase space by half.

$$x'^{\mu} = \begin{cases} q'^{a} - \theta^{ab} p_b, & \mu=a, \\ \\ \bar{q}'^{i}, & \mu=i \end{cases}$$

and

$$\pi_{\mu} = \begin{cases} p_a, & \mu=a, \\ \\ \bar{p}_i - 2\kappa B_{ij} \bar{q}'^{j}, & \mu=i. \end{cases}$$

Noncommutativity of the effective variables

- In $N-$subspace, the coordinates do not commute

$$^\star\{x^a(\sigma), x^b(\bar\sigma)\} = 2\theta^{ab}\theta(\sigma+\bar\sigma)$$

while in the $D-$subspace the momenta do not commute

$$^\star\{\pi_i(\sigma), \pi_j(\bar\sigma)\} = 4\kappa B_{ij}\delta'(\sigma+\bar\sigma)$$

Solution of T-dual constraints

- Separating dual variables into odd and even parts

$$y_\mu = k_\mu + \bar{k}_\mu, \qquad {}^\star\pi^\mu = {}^\star p^\mu + {}^\star\bar{p}^\mu$$

we obtain

$$y'_\mu = \begin{cases} \bar{k}'_a, & \mu=a, \\ \\ k'_i - \frac{2}{\kappa} B_{ij}{}^\star p^j, & \mu=i \end{cases}$$

and

$$ {}^\star\pi^\mu = \begin{cases} {}^\star\bar{p}^a - \kappa^2 \theta^{ab} \bar{k}'_b, & \mu=a, \\ \\ {}^\star p^i, & \mu=i. \end{cases}$$

Noncommutativity of T-dual effective variables

- Coordinates in the $N-$subspace are not commutative

$$^{\star}\{y_i(\sigma), y_j(\bar{\sigma})\} = \frac{4}{\kappa} B_{ij}\theta(\sigma+\bar{\sigma}) = 2^{\star}\theta_{ij}\theta(\sigma+\bar{\sigma})$$

The momenta in D-subspace are noncommutative

$$^{\star}\{^{\star}\pi^{a}(\sigma), ^{\star}\pi^{b}(\bar{\sigma})\} = 2\kappa^2\theta^{ab}\delta'(\sigma+\bar{\sigma}) = 4\kappa\, ^{\star}B^{ab}\delta'(\sigma+\bar{\sigma})$$

So, N and D-sectors of the initial and T-dual theories replace their characteristics

Effective theories

- If we substitute the solution of the boundary conditions into the canonical hamiltonians, we will obtain the effective hamiltonians. Using the equations of motion for momenta, we will find the corresponding effective lagrangians
- Effective hamiltonians

$$\mathcal{H}^{eff} = \mathcal{H}_N^{eff}(q^a, p_a) + \mathcal{H}_D^{eff}(\bar{q}^i, \bar{p}_i)$$

where

$$\mathcal{H}_N^{eff}(q^a, p_a) = \frac{\kappa}{2} q'^a G^E_{ab} q'^b + \frac{1}{2\kappa} p_a (G_E^{-1})^{ab} p_b,$$
$$\mathcal{H}_D^{eff}(\bar{q}^i, \bar{p}_i) = \frac{\kappa}{2} \bar{q}'^i G_{ij} \bar{q}'^j + \frac{1}{2\kappa} \bar{p}_i (G^{-1})^{ij} \bar{p}_j$$

Effective T-dual hamiltonian

$$^\star\mathcal{H}^{eff} = {}^\star\mathcal{H}_D^{eff}(\bar{k}_a, {}^\star\bar{p}^a) + {}^\star\mathcal{H}_N^{eff}(k_i, {}^\star p^i)$$

where

$$^\star\mathcal{H}_D^{eff}(\bar{k}_a, {}^\star\bar{p}^a) = \frac{\kappa}{2}\,\bar{k}'_a(G_E^{-1})^{ab}\bar{k}'_b + \frac{1}{2\kappa}{}^\star\bar{p}^a(G_E)_{ab}{}^\star\bar{p}^b,$$

$$^\star\mathcal{H}_N^{eff}(k_i, {}^\star p^i) = \frac{\kappa}{2}\,k'_i(G^{-1})^{ij}k'_j + \frac{1}{2\kappa}{}^\star p^i G_{ij}{}^\star p^j$$

Effective Lagrangians

- The lagrangians of the effective theories

$$\mathcal{L}^{eff} = \mathcal{L}_N(q,p) + \mathcal{L}_D(\bar{q},\bar{p}),$$
$$^\star\mathcal{L}^{eff} = {}^\star\mathcal{L}_D(\bar{k},{}^\star\bar{p}) + {}^\star\mathcal{L}_N(k,{}^\star p)$$

with

$$\mathcal{L}_N(q,p) = p_a\dot{q}^a - \mathcal{H}_N^{eff}(q^a,p_a)$$

Effective Lagrangians 2

- The explicit forms of the effective lagrangians are found by eliminating momenta using the equations of motion

$$p_a = \kappa G^E_{ab}\dot{q}^b, \quad \bar{p}_i = \kappa G_{ij}\dot{\bar{q}}^j$$

and

$$^\star\bar{p}^a = \kappa (G_E^{-1})^{ab}\dot{\bar{k}}_b, \quad ^\star p^i = \kappa (G^{-1})^{ij}\dot{k}_j$$

$$\mathcal{L}^{eff} = \mathcal{L}_N(q) + \mathcal{L}_D(\bar{p}),$$
$$^\star\mathcal{L}^{eff} = {}^\star\mathcal{L}_D(\bar{k}) + {}^\star\mathcal{L}_N(k)$$

where the lagrangians reduced to

$$\mathcal{L}_N(q) = \frac{\kappa}{2} G^E_{ab}\,\eta^{\alpha\beta}\partial_\alpha q^a \partial_\beta q^b, \qquad \mathcal{L}_D(\bar{q}) = \frac{\kappa}{2} G_{ij}\,\eta^{\alpha\beta}\partial_\alpha \bar{q}^i \partial_\beta \bar{q}^j,$$

$$^\star\mathcal{L}_D(\bar{k}) = \frac{\kappa}{2}(G_E^{-1})^{ab}\,\eta^{\alpha\beta}\partial_\alpha \bar{k}_a \partial_\beta \bar{k}_b, \qquad ^\star\mathcal{L}_N(k) = \frac{\kappa}{2}(G^{-1})^{ij}\,\eta^{\alpha\beta}\partial_\alpha k_i \partial_\beta k_j$$

T-duality between effective theories

- Let us now introduce coordinates

$$Q^\mu = \begin{bmatrix} q^a \\ \bar{q}^i \end{bmatrix}, \quad K_\mu = \begin{bmatrix} \bar{k}_a \\ k_i \end{bmatrix}$$

and effective metric

$$G^{eff}_{\mu\nu} = \begin{pmatrix} G^E_{ab} & 0 \\ 0 & G_{ij} \end{pmatrix}, \qquad {}^\star G^{\mu\nu}_{eff} = \begin{pmatrix} (G_E^{-1})^{ab} & 0 \\ 0 & (G^{-1})^{ij} \end{pmatrix}$$

T-duality between effective theories 2

- Corresponding lagrangians

$$\mathcal{L}^{eff} = \frac{\kappa}{2}\,\eta^{\alpha\beta}\,\partial_\alpha Q^\mu\; G^{eff}_{\mu\nu}\,\partial_\beta Q^\nu,$$
$${}^\star\mathcal{L}^{eff} = \frac{\kappa}{2}\,\eta^{\alpha\beta}\,\partial_\alpha K_\mu\,{}^\star G^{\mu\nu}_{eff}\,\partial_\beta K_\nu$$

- From T-duality relations for initial variables x^μ and y_μ we find

$$\partial_\pm K_\mu \cong \pm G^{eff}_{\mu\nu}\partial_\pm Q^\nu$$

This is the T-dual effective coordinate transformation law

T-duality between effective theories 3

- In absence of the effective Kalb-Ramond field T-dual metric should be inverse to the initial metric

$$(G^{eff}_{\mu\nu})^{-1} = \begin{pmatrix} G^E_{ab} & 0 \\ 0 & G_{ij} \end{pmatrix}^{-1} = \begin{pmatrix} (G_E^{-1})^{ab} & 0 \\ 0 & (G^{-1})^{ij} \end{pmatrix} = {}^\star G^{\mu\nu}_{eff}$$

- We can conclude that the effective lagrangians are T-dual to each other

Conclusion

So, we confirmed that two procedures, the T-dualization procedure and the solving of the mixed boundary conditions, treated as constraints in the Dirac consistency procedure, do commute