Black holes in the theory with nonminimal derivative coupling and some aspects of their thermodynamics

Mykola Stetsko

Department for Theoretical Physics Ivan Franko National University of Lviv

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Outline

Gravity with nonminimal derivative coupling

Motivation:(Lovelock Theory, Horndeski Gravity, Galileon Theory, String Theory).
Lovelock theory (D. Lovelock, J. Math. Phys. **12**, 498, (1971)). Lovelock Lagrangian

$$L = \sum_{h=0}^{k} c_h L_h, \tag{1}$$

where:

$$L_h = \delta^{A_1 A_2 \ldots A_{2h}}_{B_1 B_2 \ldots B_{2h}} R_{A_1 A_2}{}^{B_1 B_2} \ldots R_{A_{2h-1} A_{2h}}{}^{B_{2h-1} B_{2h}} \tag{2}$$

where $\delta^{A_1 A_2 \ldots A_h}_{B_1 B_2 \ldots B_h}$ is the generalized Kronecker delta and $R_{A_1 A_2}{}^{B_1 B_2}$ denotes Riemann tensor.
Horndeski Gravity (1974)(Galileon theory (2009))(G. W. Horndeski, Int. J. Theor. Phys. **10**, 363, (1974), C. Deffayet, S. Deser, G. Esposito-Farese, Phys. Rev. D **80**, 064015, (2009)).

Lagrangian (Horndeski, Galileon theory):

$$L = K(\varphi,\rho) - G_1(\varphi,\rho)\nabla^2\varphi + G_2(\varphi,\rho)R + G_2'((\nabla^2\varphi)^2 - (\nabla_\mu\nabla_\nu\varphi)^2) +$$
$$G_3(\varphi,\rho)G_{\mu\nu}\nabla^\mu\varphi\nabla^\nu\varphi - \frac{G_3'}{6}((\nabla^2\varphi)^3 - 3\nabla^2\varphi(\nabla_\mu\nabla_\nu\varphi)^2 + 2(\nabla_\mu\nabla_\nu\varphi)^3), \quad (3)$$

and here $\rho = \nabla_\mu\varphi\nabla^\mu\varphi$, and $F' = \frac{dF}{d\rho}$.
Dimensional reduction (String Theory):

$$ds^2_{4+n} = d\bar{s}^2_4 + e^\varphi d\tilde{y}^2_n \quad (4)$$

Four-dimensional effective action for EGB-theory can be written in the form (C. Charmousis, 2014):

$$\bar{S}_4 = \int d^4x\sqrt{-g}e^{\frac{n\varphi}{2}}\Big[\bar{R} - 2\Lambda + \alpha\bar{G} + \frac{n(n-1)}{4}(\nabla\varphi)^2 - \alpha n(n-1)\times$$
$$\bar{G}^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - \frac{\alpha}{4}n(n-1)(n-2)(\nabla\varphi)^2\nabla^2\varphi + \frac{\alpha}{16}n(n-1)^2(n-2)\times$$
$$(\nabla\varphi)^4 + e^{-\varphi}\tilde{R}\left(1 + \alpha\bar{R} + 4\alpha(n-2)(n-3)(\nabla\varphi)^2\right) + \alpha e^{-2\varphi}\tilde{G}\Big]. \quad (5)$$

Action for multidimensional gravity with nonminimal derivative coupling:

$$S = \frac{1}{16\pi}\int d^{n+1}x\sqrt{-g}\left(R - 2\Lambda - \frac{1}{2}\left(\alpha g^{\mu\nu} - \eta G^{\mu\nu}\right)\partial_\mu\varphi\partial_\nu\varphi + \mathcal{L}_m\right) \tag{6}$$

Field equations for the action (6):

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{2}(\alpha T^{(1)}_{\mu\nu} + \eta T^{(2)}_{\mu\nu}) + T^{(3)}_{\mu\nu}, \tag{7}$$

where

$$T^{(1)}_{\mu\nu} = \nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}g_{\mu\nu}\nabla^\lambda\varphi\nabla_\lambda\varphi, \tag{8}$$

$$\begin{aligned} T^{(2)}_{\mu\nu} = {} & \frac{1}{2}\nabla_\mu\varphi\nabla_\nu\varphi R - 2\nabla^\lambda\varphi\nabla_\nu\varphi R_{\lambda\mu} + \frac{1}{2}\nabla^\lambda\varphi\nabla_\lambda\varphi G_{\mu\nu} \\ & -g_{\mu\nu}\left(-\frac{1}{2}\nabla_\lambda\nabla_\kappa\varphi\nabla^\lambda\nabla^\kappa\varphi + \frac{1}{2}(\nabla^2\varphi)^2 - R_{\lambda\kappa}\nabla^\lambda\varphi\nabla^\kappa\varphi\right) \\ & -\nabla_\mu\nabla^\lambda\varphi\nabla_\nu\nabla_\lambda\varphi + \nabla_\mu\nabla_\nu\varphi\nabla^2\varphi - R_{\lambda\mu\kappa\nu}\nabla^\lambda\varphi\nabla^\kappa\varphi. \end{aligned} \tag{9}$$

Here we consider two types of Lagrangian of material field $\mathcal{L}_m$:

$$\mathcal{L}_m = \begin{cases} (-F_{\mu\nu}F^{\mu\nu})^p, & PMI \\ 4\beta^2\left(1-\sqrt{1+\frac{F_{\mu\nu}F^{\mu\nu}}{2\beta^2}}\right), & BI \end{cases} \tag{10}$$

The stress-energy tensor for corresponding material fields can be written easily. Namely for power-law nonlinear field it takes the form:

$$T^{(3)}_{\mu\nu} = \frac{g_{\mu\nu}}{2}\left(-F_{\lambda\kappa}F^{\lambda\kappa}\right)^p + 2p\left(-F_{\lambda\kappa}F^{\lambda\kappa}\right)^{p-1}F_{\mu\rho}F_{\nu}{}^{\rho}, \quad PMI. \tag{11}$$

and for Born-Infeld type of Lagrangian we derive:

$$T^{(3)}_{\mu\nu} = 2\beta^2 g_{\mu\nu}\left(1-\sqrt{1+\frac{F_{\kappa\lambda}F^{\kappa\lambda}}{2\beta^2}}\right) + \frac{2F_{\mu\rho}F_{\nu}{}^{\rho}}{\sqrt{1+\frac{F_{\kappa\lambda}F^{\kappa\lambda}}{2\beta^2}}}, \quad BI. \tag{12}$$

Black hole solution

For the scalar and electromagnetic fields we derive:

$$(\alpha g_{\mu\nu} - \eta G_{\mu\nu})\nabla^{\mu}\nabla^{\nu}\varphi = 0. \qquad (13)$$

$$\nabla_{\mu}\left(-\mathcal{F}^{p-1}F^{\mu\nu})\right) = 0, \quad PMI \qquad (14)$$

$$\nabla_{\mu}\left(\frac{F^{\mu\nu}}{\sqrt{1+\mathcal{F}/2\beta^2}}\right) = 0, \quad BI \qquad (15)$$

where $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$. Black hole's metric is supposed to take the form:

$$ds^2 = -U(r)dt^2 + W(r)dr^2 + r^2 d\Omega^{2(\varepsilon)}_{(n-1)}, \qquad (16)$$

where $d\Omega^{2(\varepsilon)}_{n-1}$ is supposed to be written as follows:

$$d\Omega^{2(\varepsilon)}_{(n-1)} = \begin{cases} d\theta^2 + \sin^2\theta d\Omega^2_{(n-2)}, & \varepsilon = 1, \\ d\theta^2 + \theta^2 d\Omega^2_{(n-2)}, & \varepsilon = 0, \\ d\theta^2 + \sinh^2\theta d\Omega^2_{(n-2)}, & \varepsilon = -1, \end{cases} \qquad (17)$$

and here $d\Omega^2_{(n-2)}$ is the line element of a $n-2$ – dimensional sphere.

Having used the evident form of the metric (16) and the equation (13) we obtain:

$$\sqrt{\frac{U}{W}}r^{n-1}\left[\alpha-\eta\frac{(n-1)}{2rW}\left(\frac{U'}{U}-\frac{(n-2)}{r}(W-1)\right)\right]\varphi'=C, \qquad (18)$$

where C is a constant of integration and $f'=\frac{\partial f}{\partial r}$. In what follows we take $C=0$ and it simplifies the procedure of integration of equations of motion. Supposing that $A=A_0(r)dt$ one can obtain a solution of the Maxwell equations (14) in the form:

$$F_{rt}=\frac{q}{r^{(n-1)/(2p-1)}}\sqrt{UW}, \quad PMI \qquad (19)$$

$$F_{rt}=\frac{q\beta}{\sqrt{q^2+\beta^2r^{2(n-1)}}}\sqrt{UW}. \quad BI \qquad (20)$$

From the equations of motion (7) we obtain:

$$(\varphi')^2=-\frac{4r^2W}{\eta(2\alpha r^2+\varepsilon\eta(n-1)(n-2))}\left(\alpha+\Lambda\eta+\frac{2^{p-1}(2p-1)\eta q^{2p}}{r^{2p(n-1)/(2p-1)}}\right) \qquad (21)$$

$$UW = \frac{\left((\alpha - \Lambda\eta)r^2 + \varepsilon\eta(n-1)(n-2) - 2^{p-1}\eta(2p-1)q^{2p}r^{2(1-\frac{p(n-1)}{2p-1})}\right)^2}{(2\alpha r^2 + \varepsilon\eta(n-1)(n-2))^2} \quad (22)$$

The metric function U can be represented in the form:

$$U(r) = \varepsilon - \frac{\mu}{r^{n-2}} - \frac{2\Lambda}{n(n-1)}r^2 - 2^p\frac{(2p-1)^2q^{2p}}{(n-1)(2p-n)}r^{2\left(1-\frac{p(n-1)}{2p-1}\right)} + \frac{(\alpha+\Lambda\eta)^2}{2\alpha\eta(n-1)r^{n-2}}\int\frac{r^{n+1}}{r^2+d^2}dr + 2^{p-1}\frac{(2p-1)(\alpha+\Lambda\eta)q^{2p}}{\alpha(n-1)r^{n-2}} \times \int\frac{r^{n+1-\frac{2p(n-1)}{2p-1}}}{r^2+d^2}dr + 2^{2p-3}\frac{(2p-1)^2\eta q^{4p}}{\alpha(n-1)r^{n-2}}\int\frac{r^{n+1-\frac{4p(n-1)}{2p-1}}}{r^2+d^2}dr \quad (23)$$

and here $d^2 = \frac{\varepsilon\eta(n-1)(n-2)}{2\alpha}$

For Born-Infeld case we obtain:

$$(\varphi')^2 = -\frac{4r^2 W\left(\alpha + \Lambda\eta - 2\beta^2\eta + \frac{2\beta\eta}{r^{n-1}}\sqrt{q^2 + \beta^2 r^{2(n-1)}}\right)}{\eta(2\alpha r^2 + \varepsilon\eta(n-1)(n-2))}; \qquad (24)$$

$$UW = \frac{\left((\alpha - \Lambda\eta + 2\beta^2\eta)r^2 + \varepsilon\eta(n-1)(n-2) - \frac{2\beta\eta}{r^{n-3}}\sqrt{q^2 + \beta^2 r^{2(n-1)}}\right)^2}{(2\alpha r^2 + \varepsilon\eta(n-1)(n-2))^2}; \qquad (25)$$

$$U(r) = \varepsilon - \frac{\mu}{r^{n-2}} - \frac{2(\Lambda - 2\beta^2)}{n(n-1)}r^2 - \frac{2\beta(\alpha - \Lambda\eta + 2\beta^2\eta)}{\alpha(n-1)r^{n-2}} \times \int\sqrt{q^2 + \beta^2 r^{2(n-1)}}dr + \frac{(\alpha + \Lambda\eta - 2\beta^2\eta)^2 + 4\beta^4\eta^2}{2\alpha\eta(n-1)\alpha\eta r^{n-2}}\int\frac{r^{n+1}dr}{r^2 + d^2} - \frac{2\beta(\alpha + \Lambda\eta - 2\beta^2\eta)d^2}{\alpha(n-1)r^{n-2}}\int\frac{\sqrt{q^2 + \beta^2 r^{2(n-1)}}dr}{r^2 + d^2} + \frac{2\beta^2\eta q^2}{\alpha(n-1)r^{n-2}}\int\frac{r^{3-n}dr}{r^2 + d^2} \qquad (26$$

Having calculated the integrals in (23) we can write:

$$U(r) = \varepsilon - \frac{\mu}{r^{n-2}} - \frac{2\Lambda}{n(n-1)}r^2 + \frac{(\alpha+\Lambda\eta)^2}{2\alpha\eta(n-1)}\left[(-1)^{\frac{n+1}{2}}\frac{d^n}{r^{n-2}}\arctan\left(\frac{r}{d}\right) + \sum_{j=0}^{\frac{n-1}{2}}(-1)^j d^{2j}\frac{r^{2(1-j)}}{n-2j}\right] - \frac{2^p(2p-1)^2q^{2p}}{(2p-n)(n-1)}r^{\frac{2(3p-pn-1)}{2p-1}} + U_{PMI}(r), (27)$$

for odd n noninteger $\frac{2p(n-1)}{2p-1}$ and d^2 is supposed to be positive and the function $U_{PMI}(r)$ takes the form:

$$U_{PMI}(r) = \frac{2^{p-1}(2p-1)^2(\alpha+\Lambda\eta)q^{2p}}{\alpha(n-1)(2p-n)}r^{2\left(1-\frac{p(n-1)}{2p-1}\right)} \times {}_2F_1\left(1, \frac{p(n-1)}{2p-1}-\frac{n}{2}; \frac{p(n-1)}{2p-1}-\frac{n}{2}+1; -\frac{d^2}{r^2}\right) + 2^{2p-3}\frac{(2p-1)^3\eta q^{4p}}{\alpha(n-1)(4p-2pn-n)}r^{2\left(1-\frac{2p(n-1)}{2p-1}\right)} \times {}_2F_1\left(1, \frac{2p(n-1)}{2p-1}-\frac{n}{2}; \frac{2p(n-1)}{2p-1}-\frac{n}{2}+1; -\frac{d^2}{r^2}\right) \quad (28)$$

For even n we obtain:

$$U(r)=\varepsilon-\frac{\mu}{r^{n-2}}-\frac{2\Lambda}{n(n-1)}r^2+\frac{(\alpha+\Lambda\eta)^2}{2\alpha\eta(n-1)}\left[(-1)^{\frac{n}{2}}\frac{d^n}{2r^{n-2}}\ln\left(\frac{r^2}{d^2}+1\right)\right.$$
$$\left.+\sum_{j=0}^{\frac{n}{2}-1}(-1)^j d^{2j}\frac{r^{2(1-j)}}{n-2j}\right]-\frac{2^p(2p-1)^2q^{2p}}{(2p-n)(n-1)}r^{\frac{2(3p-pn-1)}{2p-1}}+U_{PMI}(r).(29$$

For Born-Infeld case we can write (odd n):

$$U(r)=\varepsilon-\frac{\mu}{r^{n-2}}-\frac{2(\Lambda-2\beta^2)}{n(n-1)}r^2-\frac{2\beta^2(\alpha-\Lambda\eta+2\beta^2\eta)}{\alpha n(n-1)}r^2\times$$
$${}_2F_1\left(-\frac{1}{2},\frac{n}{2(1-n)};\frac{2-n}{2(1-n)};-\frac{q^2}{\beta^2}r^{2(1-n)}\right)+\frac{(\alpha+\Lambda\eta-2\beta^2\eta)^2+4\beta^4\eta^2}{2\alpha\eta(n-1)}$$
$$\times\left[\sum_{j=0}^{\frac{n-1}{2}}(-1)^j\frac{d^{2j}r^{2(1-j)}}{n-2j}+\frac{(-1)^{\frac{n+1}{2}}d^n}{r^{n-2}}\arctan\left(\frac{r}{d}\right)\right]+U^{(o)}_{BI}(r),(30$$

where

$$U_{BI}^{(o)}(r) = \frac{2\beta^2\eta q^2}{\alpha(n-1)}\left[\sum_{j=0}^{\frac{n-5}{2}}\frac{(-1)^j r^{6-2n+2j}}{(4-n+2j)d^{2(j+1)}} + \frac{(-1)^{\frac{n-3}{2}}}{d^{n-2}r^{n-2}}\arctan\left(\frac{r}{d}\right)\right]$$

$$-\frac{2\beta^2(\alpha+\Lambda\eta-2\beta^2\eta)d^2}{\alpha(n-1)}\sum_{j=0}^{+\infty}\frac{(-1)^j}{n-2(j+1)}\left(\frac{d}{r}\right)^{2j}\times$$

$${}_2F_1\left(-\frac{1}{2},\frac{n-2(j+1)}{2(1-n)};-\frac{n+2j}{2(1-n)};-\frac{q^2}{\beta^2}r^{2(1-n)}\right);$$

and for even n we obtain:

$$U(r) = \varepsilon - \frac{\mu}{r^{n-2}} - \frac{2(\Lambda-2\beta^2)}{n(n-1)}r^2 - \frac{2\beta^2(\alpha-\Lambda\eta+2\beta^2\eta)}{\alpha n(n-1)}r^2\times$$

$${}_2F_1\left(-\frac{1}{2},\frac{n}{2(1-n)};\frac{2-n}{2(1-n)};-\frac{q^2}{\beta^2}r^{2(1-n)}\right) + \frac{(\alpha+\Lambda\eta-2\beta^2\eta)^2+4\beta^4\eta^2}{2\alpha\eta(n-1)}$$

$$\times\left[\sum_{j=0}^{(n-2)/2}(-1)^j\frac{d^{2j}r^{2(1-j)}}{n-2j} + \frac{(-1)^{\frac{n}{2}}d^n}{2r^{n-2}}\ln\left(\frac{r^2}{d^2}+1\right)\right] + U_{BI}^{(e)}(r),(31$$

and here

$$U_{BI}^{(e)}(r)=\frac{2\beta^2\eta q^2}{\alpha(n-1)}\left[\sum_{j=0}^{\frac{n-6}{2}}\frac{(-1)^j r^{6-2n+2j}}{(4-n+2j)d^{2(j+1)}}+\frac{(-1)^{\frac{n-2}{2}}}{2d^{n-2}r^{n-2}}\ln\left(1+\frac{d^2}{r^2}\right)\right]$$

$$-\frac{2\beta^2(\alpha+\Lambda\eta-2\beta^2\eta)d^2}{\alpha(n-1)}\left[\sum_{j=0}^{+\infty}\frac{(-1)^j}{n-2(j+1)}\left(\frac{d}{r}\right)^{2j}\times\right.$$

$${}_2F_1\left(-\frac{1}{2},\frac{n-2(j+1)}{2(1-n)};-\frac{n+2j}{2(1-n)};-\frac{q^2}{\beta^2}r^{2(1-n)}\right)+(-1)^{\frac{n}{2}}\frac{d^{n-2}}{r^{n-2}}\times$$

$$\left.\left(\sum_{j=1}^{+\infty}\frac{(-1)^j}{j!}\left(-\frac{1}{2}\right)_j\left(\frac{q}{\beta}\right)^{2j}\frac{r^{2(1-n)j}}{2(n-1)j}-\ln\left(\frac{r}{d}\right)\right)\right];$$

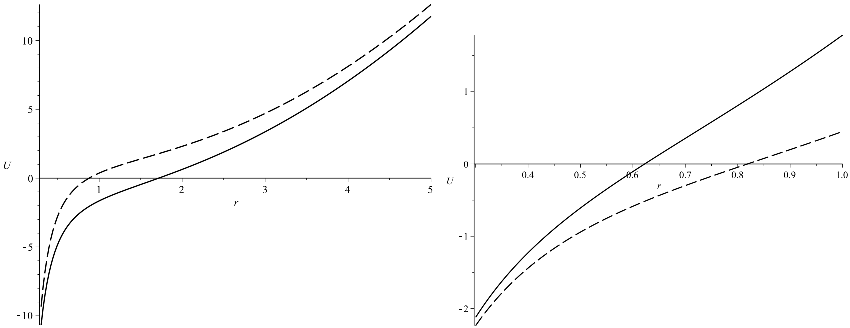

Figure: Metric functions $U(r)$ for PMI (left) and BI (right) cases

Peculiarities of the metrics for different distances

Firstly, considering large r all types show AdS-behaviour:

$$U \simeq \frac{(\alpha - \Lambda\eta)^2}{2\alpha\eta n(n-1)} r^2, \tag{32}$$

In contrast, for very small values of r the behaviour of the metric function depends on the types of material field, horizon geometry and dimension, but in all the cases it is singular.

To characterize the behaviour of BH metrics we use Kretschmann scalar:

$$R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} = \frac{1}{UW}\left(\frac{d}{dr}\left[\frac{U'}{\sqrt{UW}}\right]\right)^2 + \frac{(n-1)}{r^2W^2} \times \left(\frac{(U')^2}{U^2} + \frac{(W')^2}{W^2}\right) + \frac{2(n-1)(n-2)}{r^4W^2}(\varepsilon W - 1)^2. \tag{33}$$

Due to the fact that when $r \to \infty$ the metric function $U(r)$ behaves as (32), thus we find:

$$R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} \sim \frac{8(n+1)\alpha^2}{n(n-1)^2\eta^2}.$$

When $r \to 0$ we arrive at (PMI):

$$R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} \sim f_1(p,n)\frac{1}{r^4}.$$

For BI case we have different expressions for different values of n and ε again. Namely, when $n > 4$, $\varepsilon \neq 0$ and $n \geqslant 3$, $\varepsilon = 0$ we arrive at:

$$R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} \simeq f_2(n)\frac{1}{r^4}, \qquad (34)$$

for $n = 4$ and $n = 3$ when $\varepsilon \neq 0$ we obtain correspondingly:

$$R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} \simeq \frac{1}{r^4}\ln^2\left(1+\frac{d^2}{r^2}\right), \quad R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} \simeq \frac{\mu^2}{r^6}. \qquad (35)$$

Thermodynamics of the black holes

Temperature of the black holes can be written in the form:

$$T = \frac{\kappa}{2\pi} = \frac{1}{4\pi}\frac{U'(r_+)}{\sqrt{U(r_+)W(r_+)}} \tag{36}$$

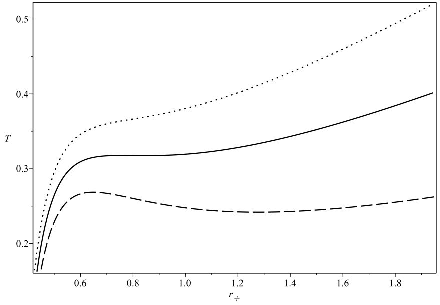

Figure: Black hole temperature for PMI case

Entropy of the black hole

Variating the action (6) we obtain the surface terms of the form (R. M. Wald, Phys. Rev. D **48**, 3427 (1993)):

$$J^{\mu} = 2\frac{\partial L}{\partial R_{\kappa\lambda\mu\nu}}\nabla_{\lambda}\delta g_{\kappa\nu} - 2\nabla_{\nu}\frac{\partial L}{\partial R_{\kappa\mu\nu\lambda}}\delta g_{\kappa\lambda} + \frac{\partial L}{\partial(\nabla_{\mu}\varphi)}\delta\varphi + \frac{\partial L}{\partial(\nabla_{\mu}A_{\nu})}\delta A_{\nu}. \quad (37)$$

Wald procedure allows to obtain 1-form, namely $J_{(1)} = J_{\mu}dx^{\mu}$ and its Hodge dual can be written:

$$\Theta_{(n)} = *J_{(1)} \quad (38)$$

Then it is supposed that an infinitesimal diffeomorphism is performed $(\delta x^{\mu} = \xi^{\mu})$. As a result one can write:

$$J_{(n)} = \Theta_{(n)} - i_{\xi} * L_{(0)} = -d * J_{(2)}, \quad (39)$$

where is assumed that the equations of motions are satisfied. Now the form $*J_{(2)}$ can be identified with $(n-1)$-form, namely $Q_{(n-1)} \equiv *J_{(2)}$.

Variation of the Hamiltonian can be written as follows:

$$\delta\mathcal{H} = \delta\int_c J_{(n)} - \int_c d\left(i_\xi\Theta_{(n)}\right) = \int_{\Sigma^{n-1}} \delta Q_{(n-1)} - i_\xi\Theta_{(n)} \qquad (40)$$

The first law can be obtained from the equation :

$$\delta\mathcal{H}_\infty = \delta\mathcal{H}_+ \qquad (41)$$

And after calculations we obtain (for PMI):

$$(\delta Q - i_\xi\Theta)_t = r^{n-1}\sqrt{UW}\left(\frac{(n-1)}{rW^2}\left(1+\frac{\eta}{4}\frac{(\varphi')^2}{W}\right)\delta W + \frac{2^p p(2p-1)}{(UW)^p}\right.$$
$$\left.(\psi')^{2(p-1)}\psi\left(\psi'\left(\frac{\delta W}{W}+\frac{\delta U}{U}\right)-2\delta\psi'\right) - \frac{\eta(n-1)}{2rW}\delta\left(\frac{(\varphi')^2}{W}\right)\right)\Omega_{(n-1)}. \qquad (42)$$

At the infinity we arrive at (PMI):

$$(\delta Q - i_\xi\Theta)_t = (n-1)\delta\mu - 4p(2p-1)2^{p-1}q^{2(p-1)}\psi_0\delta q. \qquad (43)$$

and for BI case we derive

$$(\delta Q - i_\xi\Theta)_t = (n-1)\delta\mu - 4\psi_0\delta q \qquad (44)$$

And as a result we can write:

$$\delta\mathcal{H}_\infty = \delta M - \Phi_q \delta Q, \tag{45}$$

where we have:

$$M = \frac{(n-1)\omega_{n-1}}{16\pi}\mu, \tag{46}$$

$$Q = \frac{\omega_{n-1}}{4\pi} 2^{p-1} q^{2p-1}, \quad (PMI) \quad Q = \frac{\omega_{n-1}}{4\pi} q \quad (BI) \tag{47}$$

At the horizon we obtain:

$$\delta\mathcal{H}_+ = \frac{(n-1)\omega_{n-1}}{16\pi} U'(r_+) r_+^{n-2} \delta r_+ = \left(1 + \frac{\eta}{4} \frac{(\varphi')^2}{W}\bigg|_{r_+}\right) T\delta\left(\frac{\mathcal{A}}{4}\right). \tag{48}$$

where $\mathcal{A} = \omega_{n-1} r_+^{n-1}$ is the horizon area of the black hole.
Definition of black hole entropy is disputed, but nevertheless we can write:

$$\delta M = \left(1 + \frac{\eta}{4} \frac{(\varphi')^2}{W}\bigg|_{r_+}\right) T\delta\left(\frac{\mathcal{A}}{4}\right) + \Phi_q \delta Q. \tag{49}$$

Alternative way to derive relation for the entropy is based on euclidean formulation:

$$S_{tot} = S_{EH} + S_{GHY} + S_{ct} \tag{50}$$

Boundary term S_{GHY} can be written in the form:

$$S^{(nm)}_{GHY} = \frac{1}{8\pi}\int d^n x\sqrt{|h|}\left(K + \frac{\eta}{4}\left[\nabla^\mu\varphi\nabla^\nu\varphi K_{\mu\nu} + (\partial_n\varphi\partial_n\varphi + (\nabla\varphi)^2)K\right]\right). (5$$

Counterterms can be constructed with help of Fefferman-Graham representation of the metric:

$$ds^2 = \frac{l^2}{4\rho^2}d\rho^2 + \frac{l^2}{\rho}g_{ij}dx^i dx^j; \tag{52}$$

metric tensor g_{ij} and the scalar field φ are supposed to take the form:

$$g_{ij} = g^{(0)}_{ij} + \rho g^{(2)}_{ij} + \rho^2 g^{(4)}_{ij} + \ldots; \tag{53}$$

$$\varphi = \varphi^{(0)} + \rho\varphi^{(2)} + \rho^2\varphi^{(4)} + \ldots. \tag{54}$$

Having performed some calculations we arrive at the counterterm action of the form:

$$S_{ct} = \frac{1}{8\pi}\int d^n x\sqrt{|h|}\left(\frac{2(n-1)}{l} + \frac{l}{(n-2)}\mathcal{R} - \frac{l}{2(n-2)}\left(\alpha - \frac{\eta(n-1)(n-4)}{2l^2}\right)\nabla^k\varphi\nabla_k\varphi + \dots\right), \tag{55}$$

when $\alpha \neq \eta n(n-1)/2l^2$ and here $\Lambda = -n(n-1)/2l^2$.
If $\alpha = \eta n(n-1)/2l^2$ the scalar field has additional logarithmic contribution: $\sim \varphi_c \ln\rho$, then we write:

$$S^c_{ct} = \frac{1}{8\pi}\int d^n x\sqrt{|h|}\left(\frac{2(n-1)}{l}\left(1 + \frac{\eta\varphi_c^2}{l^2}\right) + \frac{l}{(n-2)}\left(1 - \frac{\eta\varphi_c^2}{l^2}\right)\mathcal{R} - \frac{\eta(n-1)}{4l}\nabla^k\varphi\nabla_k\varphi + \dots\right), \tag{56}$$

and here $\Lambda = -n(n-1)(1 + 2\eta\varphi_c^2/l^2)/2l^2$.

Now having the euclidean action (50) we can find the entropy using the relation:

$$S = \left(\bar{\beta}\frac{\partial}{\partial\bar{\beta}} - \hat{I}\right) S_{tot}, \tag{57}$$

and here $\bar{\beta}$ denotes inverse temperature.
To define mass we use boundary stress energy tensor:

$$T_{ij} = \frac{2}{\sqrt{|h|}}\frac{\delta S_{tot}}{\delta h^{ij}}, \tag{58}$$

and as a result we can write:

$$M = \frac{1}{8\pi}\int_{\mathcal{B}} d^{n-2}x\sqrt{\sigma}\, T_{ij}\bar{n}^i\xi^j, \tag{59}$$

where $\bar{n}^i$ is a timelike unit normal to the boundary $\mathcal{B}$ and ξ^j is timelike Killing vector.
More about PMI case: M. M. Stetsko, Phys. Rev. D **99**, 044028 (2019).

Thank you for your attention