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A 3D Superconformal QM with $sl(2|1)$ dynamical symmetry

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Based on:

I.E. Cunha & F.T., preprint CBPF-NF-002/19 arXiv:1906.11705[hep-th]

Previous works (three methods):

Quantization of world-line superconformal actions ($1D$ sigma-models):
I. E. Cunha, N. L. Holanda & F.T.,
PRD (2017), arXiv:1610.07205

Symmetries of Matrix PDEs:
F.T. & M. Valenzuela,
Adv. Math. Phys. (2018), arXiv:1705.04004

Direct approach:
- N. Aizawa, Z. Kuznetsova & F.T.,
JMP (2018), arXiv:1711.02923
- N. Aizawa, I. E. Cunha, Z. Kuznetsova & F.T.,
JMP (2019), arXiv:1812.00873

F. Calogero (1969) - **sl_2-invariance, $\frac{1}{x^2}$ potential.**

de Alfaro-Fubini-Furlan (1976) - **oscillator term addition (discrete, grounded from below spectrum, ground state)**.

Conformal Mechanics in the new Millennium (motivations):

Holography: $AdS_2 - CFT_1$

test particle close to RN BH horizon (Britto-Pacumio et al. 1999).

AdS_2 holography and SYK models (Maldacena and Stanford 2016).

Contents:

- Construction of the $3D$ SCQM model
- Construction of the $3D$ β-deformed oscillator
- Determination of the $sl(2|1)$ lwr's.
- Alternative selections of Hilbert spaces

(following Miyazaki-Tsutsui '02 and Fehér-Tsutsui-Fülöp '05)
- Spectra and zigzag patterns of vacuum energies.
- Interpolating linear/quadratic regimes for energy degeneracies
- Orthonormal eigenstates from associated Laguerre polynomials and spin-spherical harmonics.
- Dimensional reductions.
- Comment on larger algebraic structures.

The $3D$ $SCQM$ model:

Natural Ansatz for $\mathcal{N}=2$ susy ($a=1,2$):

$$Q_a = \frac{1}{\sqrt{2}}\gamma_a\left(\partial\!\!\!/ - \frac{\beta}{r^2}N_F\, r\!\!\!/\right).$$

β is a real parameter, $r=\sqrt{x_1^2+x_2^2+x_3^2}$ the radial coordinate, while $\partial\!\!\!/ = \partial_i h_i$ and $r\!\!\!/ = x_i h_i$ are written in terms of quaternions (h_i); γ_a are Clifford matrices s.t. $[\gamma_a, h_i]=0$; N_F is the Fermion Parity Operator.

$\mathcal{N}=2$ supersymmetric quantum mechanics:

$$\{Q_a, Q_b\} = 2\delta_{ab}H, \qquad [H, Q_a]=0.$$

The 4×4 matrix supersymmetric Hamiltonian H is given by

$$H = \begin{pmatrix} (-\frac{1}{2}\nabla^2 + \frac{2\beta}{r^2}\overrightarrow{\mathbf{S}}\cdot\overrightarrow{\mathbf{L}} + \frac{\beta(\beta+1)}{2r^2})\mathbb{I}_2 & 0 \\ 0 & (-\frac{1}{2}\nabla^2 - \frac{2\beta}{r^2}\overrightarrow{\mathbf{S}}\cdot\overrightarrow{\mathbf{L}} + \frac{\beta(\beta-1)}{2r^2})\mathbb{I}_2 \end{pmatrix}$$

where $\nabla^2 = \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$ is the three-dimensional Laplacian, $\overrightarrow{\mathbf{S}}$ is the spin-$\frac{1}{2}$ and $\overrightarrow{\mathbf{L}}$ is a orbital angular momentum.

The Hamiltonian H is Hermitian. Since the spin is $\frac{1}{2}$, the total angular momentum $\overrightarrow{\mathbf{J}} = \overrightarrow{\mathbf{L}} + \overrightarrow{\mathbf{S}}$ of the quantum-mechanical system is half-integer.

The Hamiltonian is non-diagonal; on the other hand, due to

$$\overrightarrow{\mathbf{L}} \cdot \overrightarrow{\mathbf{S}} = \frac{1}{2}(\overrightarrow{\mathbf{J}}^2 - \overrightarrow{\mathbf{L}}^2 - \overrightarrow{\mathbf{S}}^2) = \frac{1}{2}(j(j+1) - l(l+1) - \frac{3}{4}),$$

it gets diagonalized in each sector of given total j and orbital l angular momentum.

In each such sector it corresponds to a constant kinetic term plus a diagonal potential term proportional to $\frac{1}{r^2}$.

$sl(2|1)$ superconformal algebra:

DFF construction: Introduce the conformal partner of H as the rotationally invariant operator K of scaling dimension $[K] = -1$:

$$K = \frac{1}{2} r^2 \mathbb{I}_4$$

Verify whether the repeated (anti)commutators of the operators Q_a and K close the superconformal algebra $sl(2|1)$. Itis so!.

Four extra operators $(\overline{Q}_a, D, R)$ have to be added. D is the (bosonic) dilatation operator which, together with H, K, close the $sl(2)$ subalgebra, two fermionic operators $\overline{Q}_a$ and R is the $u(1)$ R-symmetry bosonic operator of $sl(2|1)$:

$$\begin{array}{llllll}
[D,H] & = -2iH, & [D,K] & = 2iK, & [H,K] & = iD, \\
[D,Q_a] & = -iQ_a, & [D,\overline{Q}_a] & = i\overline{Q}_a, & & \\
[H,\overline{Q}_a] & = iQ_a & [K,Q_a] & = -i\overline{Q}_a, & & \\
\{Q_a,Q_b\} & = 2\delta_{ab}H, & \{\overline{Q}_a,\overline{Q}_b\} & = 2\delta_{ab}K, & \{Q_a,\overline{Q}_b\} & = \delta_{ab}D + \epsilon_{ab}R, \\
[R,Q_a] & = -3i\epsilon_{ab}Q_b, & [R,\overline{Q}_a] & = -3i\epsilon_{ab}\overline{Q}_b, & &
\end{array}$$

with the antisymmetric tensor ϵ_{ab} normalized so that $\epsilon_{12} = 1$.

Deformed oscillator:

By setting

$$H_{osc} \quad = \quad H + K,$$

we obtain the 4×4 matrix deformed oscillator Hamiltonian H_{osc} whose spectrum is discrete and bounded from below.

By construction, the $sl(2|1)$ dynamical symmetry of the H Hamiltonian acts as a spectrum-generating superalgebra for the H_{osc} Hamiltonian.

The explicit expression is

$$H_{osc} \quad = \quad -\frac{1}{2}\nabla^2 \cdot \mathbb{I}_4 + \frac{1}{2r^2}(\beta^2 \cdot \mathbb{I}_4 + \beta N_F(1 + 4 \cdot \mathbb{I}_2 \otimes \vec{\mathbf{S}} \cdot \vec{\mathbf{L}})) + \frac{1}{2}r^2 \cdot \mathbb{I}_4.$$

Appearance of two-component spherical harmonics:

$$j = l + \delta\frac{1}{2}, \qquad \text{for} \quad \delta = \pm 1.$$

In the given j, l sector we get

$$\vec{\mathbf{L}} \cdot \vec{\mathbf{S}} = \frac{1}{2}\alpha, \qquad \text{with} \quad \alpha = \delta(j + \frac{1}{2}) - 1.$$

The energy eigenstates of the system are described with the help of the two-component $\mathcal{Y}_{j,l,m}(\theta, \phi)$ spin spherical harmonics given by

$$\mathcal{Y}_{j,j-\frac{1}{2}\delta,m}(\theta,\phi) = \frac{1}{\sqrt{2j-\delta+1}} \begin{pmatrix} \delta\sqrt{j+\frac{1}{2}(1-\delta)+\delta m}\, Y^{m-\frac{1}{2}}_{j-\frac{1}{2}\delta}(\theta,\phi) \\ \sqrt{j+\frac{1}{2}(1-\delta)-\delta m}\, Y^{m+\frac{1}{2}}_{j-\frac{1}{2}\delta}(\theta,\phi) \end{pmatrix},$$

where $Y^n_l(\theta, \phi)$ (for $n = -l, -l+1, \ldots, l$) are the ordinary spherical harmonics.

The spin spherical harmonics $\mathcal{Y}_{j,j-\frac{1}{2}\delta,m}(\theta, \phi)$ are the eigenstates for the compatible observable operators $\vec{\mathbf{J}} \cdot \vec{\mathbf{J}}$, $\vec{\mathbf{L}} \cdot \vec{\mathbf{L}}$, J_z, with eigenvalues $j(j+1)$, $(j - \frac{1}{2}\delta)(j - \frac{1}{2}\delta + 1)$, m, respectively.

Creation (annihilation) operators:

$$a_b = Q_b + i\overline{Q}_b, \qquad a_b^\dagger = Q_b - i\overline{Q}_b.$$

Indeed, we obtain

$$H_{osc} = \frac{1}{2}\{a_1, a_1^\dagger\} = \frac{1}{2}\{a_2, a_2^\dagger\},$$

together with

$$[H_{osc}, a_b] = -a_b, \qquad [H_{osc}, {a_b}^\dagger] = a_b^\dagger.$$

For completeness we also present the commutators

$$[a_1, a_1^\dagger] = [a_2, a_2^\dagger] = 3 \cdot \mathbb{I}_4 + 4 \cdot \mathbb{I}_2 \otimes \vec{\mathbf{S}} \cdot \vec{\mathbf{L}} - 2\beta N_F.$$

$$a_b^\pm = \frac{\not{r}}{r\sqrt{2}}\gamma_b(\mathbb{I}_4 \cdot (\partial_r \mp r) - \frac{2}{r}\mathbb{I}_2 \otimes \vec{\mathbf{S}} \cdot \vec{\mathbf{L}} - \frac{\beta}{r}N_F).$$

They can be factorized as

$$a_b^\pm = \frac{\not{r}}{r\sqrt{2}}\gamma_b a^\pm, \qquad \text{with} \quad a^\pm = (\mathbb{I}_4 \cdot (\partial_r \mp r) - \frac{2}{r}\mathbb{I}_2 \otimes \vec{\mathbf{S}} \cdot \vec{\mathbf{L}} - \frac{\beta}{r}N_F).$$

Lowest weight vectors:

A lowest weight state Ψ_{lws} is defined to satisfy

$$a_b^- \Psi_{lws} = 0.$$

Due to the factorization, in both $b = 1, 2$ cases, this is tantamount to satisfy $a^- \Psi_{lws} = 0$.

The vectors $a_1^+ v$ and $a_2^+ v$, with v belonging to the lowest weight representation, differ by a phase.

Therefore, the action of a_1^+, a_2^+ produces the same ray vector characterizing a physical state of the Hilbert space.

We search for solutions $\Psi_{j,\delta,m}^{\epsilon}(r, \theta, \phi)$ of the form

$$\Psi_{j,\delta,m}^{\epsilon}(r,\theta,\phi) = f_{j,\delta}^{\epsilon}(r) \cdot e_{\epsilon} \otimes \mathcal{Y}_{j,j-\frac{1}{2}\delta,m}(\theta,\phi), \quad \text{with} \quad \epsilon = \pm 1.$$

The sign of ϵ (no summation over this repeated index) refers to the bosonic (fermionic) states with respective eigenvalues $\epsilon = +1$ ($\epsilon = -1$) of the Fermion Parity Operator N_F; we have $e_{+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Solutions:

Solutions are obtained for

$$f^{\epsilon}_{j,\delta}(r) \quad = \quad r^{\gamma(j,\delta,\epsilon)} e^{-\frac{1}{2}r^2},$$

where

$$\gamma_{(j,\delta,\epsilon)}(\beta) \quad = \quad \alpha + \beta\epsilon = \delta(j + \frac{1}{2}) + \beta\epsilon - 1.$$

The corresponding lowest weight state energy eigenvalue $E_{j,\delta,\epsilon}(\beta)$ from

$$H_{osc}(\beta)\Psi^{\epsilon}_{j,\delta,m}(r,\theta,\phi) \quad = \quad E_{j,\delta,\epsilon}(\beta)\Psi^{\epsilon}_{j,\delta,m}(r,\theta,\phi)$$

is

$$E_{j,\delta,\epsilon}(\beta) \quad = \quad \delta(j + \frac{1}{2}) + \beta\epsilon + \frac{1}{2}.$$

Since $E_{j,\delta,\epsilon}(\beta)$ does not depend on the quantum number m, this energy eigenvalue is $(2j+1)$ times degenerate.

Alternative Hilbert spaces

Without loss of generality we can restrict the real parameter β to belong to the half-line $\beta \geq 0$ since the mapping $\beta \leftrightarrow -\beta$ is recovered by a similarity transformation which exchanges bosons into fermions:

$$SH_{osc}(\beta)S^{-1} = H_{osc}(-\beta) \qquad \text{with} \quad S = \sigma_1 \otimes \mathbb{I}_2.$$

To the following j, δ, ϵ, m quantum numbers,

$$j \in \frac{1}{2} + \mathbb{N}_0, \qquad \delta = \pm 1, \qquad \epsilon = \pm 1, \qquad m = -j, -j+1, \ldots, j,$$

is associated an $sl(2|1)$ lowest weight vector and its induced rep.
Two choices to select the Hilbert space naturally appear:

- *case i*: the wave functions can be singular at the origin, but they need to be normalized,
- *case ii*: the wave functions are assumed to be regular at the origin.

Case i corresponds in restricting the admissible lowest weight representations to those satisfying the necessary and sufficient condition

$$2\gamma_{(j,\delta,\epsilon)}(\beta)+3 \quad > \quad 0.$$

The normalizability condition is equivalent to the requirement

$$E_{j,\delta,\epsilon}(\beta) \quad > \quad 0$$

for the lowest weight energy $E_{j,\delta,\epsilon}(\beta)$.

Case ii corresponds in restricting the admissible lowest weight representations to those satisfying the condition

$$\gamma_{(j,\delta,\epsilon)}(\beta) \quad \geq \quad 0 \qquad \text{for} \quad \beta \geq 0.$$

The single-valuedness of the wave functions at the origin implies that $\gamma_{(j,\delta,\epsilon)}(\beta)=0$ can only be realized with vanishing ($l=0$) orbital angular momentum. At $\beta=0$ one recovers the vacuum state of the undeformed oscillator.

For the deformed $\beta>0$ oscillator the strict inequality follows

$$\gamma_{(j,\delta,\epsilon)}(\beta) \quad > \quad 0 \qquad \text{for} \quad \beta>0$$

Table (up to $j=\frac{5}{2}$) of the β range of admissible lowest weight representations under *norm* (*case i*) and *reg* (*case ii*) conditions:

j	δ	ϵ	γ	E	*norm*	*reg*
$\frac{1}{2}$	$+$	$+$	β	$\frac{3}{2}+\beta$	$\beta\geq 0$	$\beta\geq 0$
$\frac{1}{2}$	$+$	$-$	$-\beta$	$\frac{3}{2}-\beta$	$0\leq\beta<\frac{3}{2}$	$\beta=0$
$\frac{1}{2}$	$-$	$+$	$\beta-2$	$-\frac{1}{2}+\beta$	$\beta>\frac{1}{2}$	$\beta>2$
$\frac{1}{2}$	$-$	$-$	$-\beta-2$	$-\frac{1}{2}-\beta$	$\times$	$\times$
$\frac{3}{2}$	$+$	$+$	$\beta+1$	$\frac{5}{2}+\beta$	$\beta\geq 0$	$\beta\geq 0$
$\frac{3}{2}$	$+$	$-$	$-\beta+1$	$\frac{5}{2}-\beta$	$0\leq\beta<\frac{5}{2}$	$0\leq\beta<1$
$\frac{3}{2}$	$-$	$+$	$\beta-3$	$-\frac{3}{2}+\beta$	$\beta>\frac{3}{2}$	$\beta>3$
$\frac{3}{2}$	$-$	$-$	$-\beta-3$	$-\frac{3}{2}-\beta$	$\times$	$\times$
$\frac{5}{2}$	$+$	$+$	$\beta+2$	$\frac{7}{2}+\beta$	$\beta\geq 0$	$\beta\geq 0$
$\frac{5}{2}$	$+$	$-$	$-\beta+2$	$\frac{7}{2}-\beta$	$0\leq\beta<\frac{7}{2}$	$0\leq\beta<2$
$\frac{5}{2}$	$-$	$+$	$\beta-4$	$-\frac{5}{2}+\beta$	$\beta>\frac{5}{2}$	$\beta>4$
$\frac{5}{2}$	$-$	$-$	$-\beta-4$	$-\frac{5}{2}-\beta$	$\times$	$\times$

For the $\beta > 0$ deformed oscillators, the Hilbert spaces $\mathcal{H}_{norm}$ and $\mathcal{H}_{reg}$ are direct sums of the lowest weight representations with $j \in \frac{1}{2} + \mathbb{N}_0$ satisfying (depending on δ, ϵ)

		$\mathcal{H}_{norm}$:	$\mathcal{H}_{reg}$:
$\delta = +1$	$\epsilon = +1$	any j	any j
$\delta = +1$	$\epsilon = -1$	$j > \beta - 1$	$j > \beta + \frac{1}{2}$
$\delta = -1$	$\epsilon = +1$	$j < \beta$	$j < \beta - \frac{3}{2}$
$\delta = -1$	$\epsilon = -1$	no j	no j

Spectrum (Hilbert space $\mathcal{H}_{norm}$)

For $\beta \geq \frac{1}{2}$ it is convenient to introduce, via the floor function, the parameter μ, defined as

$$\mu = \{\beta - \tfrac{1}{2}\} = (\beta - \tfrac{1}{2}) - \lfloor \beta - \tfrac{1}{2} \rfloor, \qquad p = \lfloor \beta - \tfrac{1}{2} \rfloor,$$
$$\text{so that} \quad \mu \in [0, 1[, \quad p \in \mathbb{N}_0 \quad \text{and} \qquad \beta = \tfrac{1}{2} + \mu + p.$$

The results for the spectrum split into six different cases which have to be separately analyzed:

- **case I**: $\beta = 0$ (the undeformed oscillator),
- **case II**: $\beta = 1 + p$, with $p \in \mathbb{N}_0$ $(p = 0, 1, 2, \ldots)$,
- **case III**: $\beta = \frac{1}{2} + p$, with $p \in \mathbb{N}_0$,
- **case IV**: $0 < \beta < \frac{1}{2}$,
- **case V**: $0 < \mu < \frac{1}{2}$, therefore $\beta = \frac{1}{2} + \mu + p$, with $p \in \mathbb{N}_0$,
- **case VI**: $\frac{1}{2} < \mu < 1$, therefore $\beta = \frac{1}{2} + \mu + p$, with $p \in \mathbb{N}_0$.

The energy eigenvalues corresponding to the above cases are

- **case I**: $E_n = \frac{3}{2} + n$, where $n \in \mathbb{N}_0$ is a non-negative integer.
  The vacuum energy is $E_{vac} = \frac{3}{2}$; the ground state is four times degenerated, with two bosonic and two fermionic eigenstates (hence "$2_B + 2_F$").
  The vacuum lowest weight vectors are specified by the quantum numbers $j = \frac{1}{2}$, $\delta = +1$, $\epsilon = \pm 1$ and (here and in the following) all compatible values $m = -j, \ldots, j$.
- **case II**: $E_n = \frac{1}{2} + n$, with $n \in \mathbb{N}_0$.
  The vacuum energy is $E_{vac} = \frac{1}{2}$; the degeneration of the ground state is $2(p+1)$, with $p+1$ bosonic and $p+1$ fermionic eigenstates, and is therefore denoted as "$(p+1)_B + (p+1)_F$".
  The vacuum lowest weight vectors are specified by $j = \frac{1}{2} + p$, with either $\delta = +1$, $\epsilon = -1$ or $\delta = -1$, $\epsilon = +1$.
- **case III**: $E_n = 1 + n$, with $n \in \mathbb{N}_0$.
  The vacuum energy is $E_{vac} = \frac{1}{2}$; the degeneration of the ground state is $4p + 2$, with $2p$ bosonic and $2(p+1)$ fermionic eigenstates, and is therefore denoted as "$(2p)_B + (2p+2)_F$".
  For $p = 0$ the two vacuum lowest vectors are specified by $j = \frac{1}{2}$, $\delta = +1$, $\epsilon = -1$.
  For $p > 0$ the vacuum lowest vectors are specified either by $j = \frac{1}{2} + p$, $\delta = +1$, $\epsilon = -1$ or by $j = p - \frac{1}{2}$, $\delta = -1$, $\epsilon = +1$.

- **case IV**: two series of energy eigenvalues $E_n^{\pm} = \frac{3}{2} \pm \beta + n$, with $n \in \mathbb{N}_0$, are encountered.
  The vacuum energy is $E_{vac} = \frac{3}{2} - \beta$; the ground state is fermionic and doubly degenerated ("2_F").
  The two vacuum lowest weight vectors are specified by $j = \frac{1}{2}$, $\delta = +1$, $\epsilon = -1$.
- **case V**: two series of energy eigenvalues $E_n^- = \mu + n$, $E_n^+ = 1 - \mu + n$, with $n \in \mathbb{N}_0$, are encountered.
  The vacuum energy is $E_{vac} = \mu$; the ground state is bosonic and $(2p+2)$-times degenerated (hence "$(2p+2)_B$").
  The vacuum lowest weight vectors are specified by $j = \frac{1}{2} + p$, $\delta = -1$, $\epsilon = +1$.
- **case VI**: two series of energy eigenvalues $E_n^- = 1 - \mu + n$, $E_n^+ = \mu + n$, with $n \in \mathbb{N}_0$, are encountered.
  The vacuum energy is $E_{vac} = 1 - \mu$; the ground state is fermionic and $(2p+2)$-times degenerated (hence "$(2p+2)_F$").
  The vacuum lowest weight vectors are specified by $j = \frac{1}{2} + p$, $\delta = +1$, $\epsilon = -1$.

Important remark. The energy spectrum of the **V** and **VI** cases coincides under a

$$\mu \quad \leftrightarrow 1-\mu, \qquad \text{with} \quad \mu \neq 0, \tfrac{1}{2},$$

duality transformation.

Under this duality transformation the parity (bosonic/fermionic) of the ground state is exchanged. On the other hand, the degeneracies of the energy eigenvalues above the ground state are not respected by the duality transformation.

Example: $\mu = \frac{1}{4}$ with $p = 0$ (dually related $\beta = \frac{3}{4}$ and $\beta = \frac{5}{4}$ cases).

The lwv's appearing in the first five energy levels are

E	$\beta = \frac{3}{4}$	$\beta = \frac{5}{4}$
$\frac{9}{4}$	$\frac{1}{2} + B$	$\frac{5}{2} + F$
$\frac{7}{4}$	$\frac{3}{2} + F$	$\times$
$\frac{5}{4}$	$\times$	$\frac{3}{2} + F$
$\frac{3}{4}$	$\frac{1}{2} + F$	$\frac{1}{2} - B$
$\frac{1}{4}$	$\frac{1}{2} - B$	$\frac{1}{2} + F$

Computation of degeneracies:

The degeneracy of each energy level is finite and can be computed recursively.. Let $n(E)$ be the total number of distinct, admissible, lwv's in the Hilbert space and let $d(E)$ be the number of degenerate eigenstates at energy level E. At energy level $E+1$ we have

$$d(E+1) \;\; = \;\; d(E) + n(E+1).$$

The $d(E)$ term in the r.h.s. gives the number of descendant states obtained by applying $a_1^\dagger$ to the degenerate states at energy E, while the $n(E+1)$ term corresponds to the number of

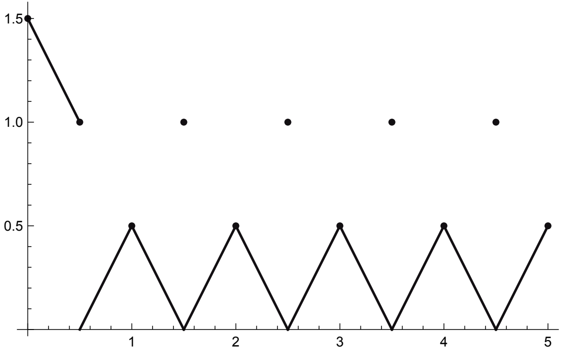
For the case above:

E	$d_{\beta=\frac{3}{4}}(E)$	$d_{\beta=\frac{5}{4}}(E)$
$\frac{9}{4}$	4	12
$\frac{7}{4}$	6	2
$\frac{5}{4}$	2	6
$\frac{3}{4}$	2	2
$\frac{1}{4}$	2	2

One can see that $\frac{5}{4}$ is the first energy level where an inequality of the degeneracies is produced

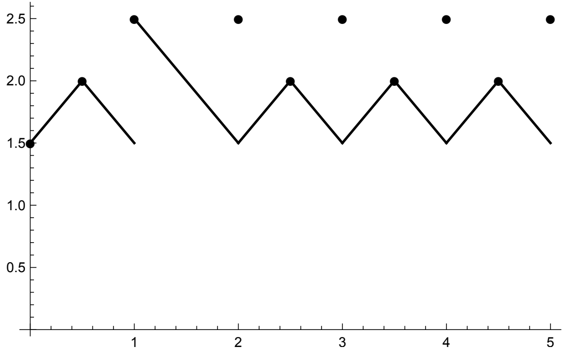
$$d_{\beta=\frac{3}{4}}(\frac{5}{4}) \neq d_{\beta=\frac{5}{4}}(\frac{5}{4}).$$

Vacuum Energy (Hilbert space I):



The vacuum energy $E_{vac}(\beta)$ of the model is portrayed in the y axis, with β up to $\beta \leq 5$ depicted in the x axis. This diagram refers to the Hilbert space admitting singular, but normalized wave functions at the origin. Starting from $\beta > \frac{1}{2}$, the graph is composed by a triangle wave of half-open line segments plus isolated points at $\beta = \frac{1}{2} + \mathbb{N}$.

Vacuum Energy (Hilbert space II):



The vacuum energy $E_{vac}(\beta)$ of the model is portrayed in the y axis, with β up to $\beta \leq 5$ depicted in the x axis. This diagram refers to the Hilbert space satisfying the condition that its wave functions are regular at the origin. For $\beta > 0$, the vacuum energy is always comprised in the interval $\frac{3}{2} < E_{vac}(\beta) \leq \frac{5}{2}$.

Degeneracy of the eigenstates:

At $\beta = 0$ H_{osc} corresponds to four copies of the ordinary isotropic three-dimensional oscillator. Its degeneracy $d_{\beta=0}(n)$ is

$$d_{\beta=0}(n) = 4 \cdot d(n), \qquad \text{with} \qquad d(n) = \frac{1}{2}(n^2 + 3n + 2).$$

Degeneracies for $\beta = \frac{1}{2} + \mathbb{N}_0$ and $\beta = 1 + \mathbb{N}_0$ with $\mathcal{H}_{norm}$ Hilbert space:

Case a: $\beta = \frac{1}{2} + p$ (energy levels $E_n = n + 1$) with $p, n \in \mathbb{N}_0$.
The degeneracy $d_{\beta=\frac{1}{2}+p}(E_n)$ grows linearly (mimicking a two-dimensional oscillator) up to $n = p$; it then grows quadratically starting from $n = p + 1$:

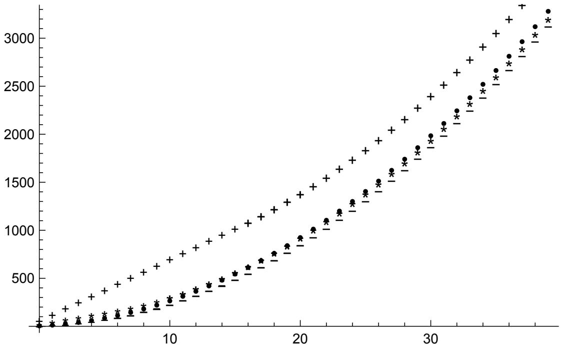
$$\begin{aligned} d_{\beta=\frac{1}{2}+p}(E_n) &= 2(n+1)(2p+1) \text{ for } n = 0, 1, 2, \ldots, p, \\ d_{\beta=\frac{1}{2}+p}(E_n) &= 2 \cdot (q^2 + 2(p+1)q + (p+1)(2p+1)) \text{ for } n = p + q \text{ with } q = 0, 1, 2, \ldots . \end{aligned}$$

Case b: $\beta = 1 + p$ (energy levels $E_n = n + \frac{1}{2}$) with $p, n \in \mathbb{N}_0$.

As in the previous case, the degeneracy $d_{\beta=1+p}(E_n)$ grows linearly (mimicking a two-dimensional oscillator) up to $n = p$; it then grows quadratically starting from $n = p + 1$:

$$\begin{aligned} d_{\beta=1+p}(E_n) &= 4(n+1)(p+1) \quad \text{for} \quad n = 0, 1, 2, \ldots, p, \\ d_{\beta=1+p}(E_n) &= 2 \cdot (q^2 + (2p+1)q + 2(p+1)^2) \quad \text{for} \quad n = p + q \quad \text{with} \quad q = 0, 1, 2, \ldots . \end{aligned}$$

Energy degeneracy at various β:



Energy degeneracy (y axis) for the $\mathcal{H}_{norm}$ Hilbert space at the integer values $\beta = 0, 2, 6, 16$. In the x axis are reported the 40 lowest energy eigenvalues. The "$\bullet$" bullet denotes the $\beta = 0$ undeformed oscillator, while "$-$", "$*$" and "$+$" stand, respectively, for the $\beta = 2, 6, 16$, cases. One can note the "bending" of the $\beta = 16$ curve around energy $E = 16$.

Orthonormal eigenstates

The excited eigenstates $(a_1^+)^k \Psi^{\epsilon}_{j,\delta,m}(r,\theta,\phi)$, obtained by applying k times the a_1^+ creation operator (1), are orthogonal.
The computation of their normalization factors which make the wave functions orthonormal involves the computation of Rodrigues-type formulas for recursive polynomials in the radial coordinate r. These recursive polynomials can be recovered from the associated Laguerre's polynomials.

$$a_1^+ \quad = \quad \frac{1}{\sqrt{2}} \gamma_1 \frac{\not{r}}{r} \left(\mathbb{I}_4 \cdot (\partial_r - r) - \frac{2}{r} \mathbb{I}_2 \otimes \vec{\mathbf{S}} \cdot \vec{\mathbf{L}} - \frac{\beta}{r} N_F \right)$$

$$\Psi^{\epsilon}_{j,\delta,m}(r,\theta,\phi) \quad = \quad e_\epsilon \otimes \mathcal{Y}_{j,j-\frac{1}{2}\delta,m}(\theta,\phi) \cdot r^{\beta\epsilon+\delta j+\frac{1}{2}\delta-1} e^{-\frac{1}{2}r^2}.$$

The action of $\frac{\not{r}}{r}$ can be read from

$$\frac{\vec{\mathbf{r}} \cdot \vec{\sigma}}{r} \mathcal{Y}_{j,j-\frac{1}{2}\delta,m}(\theta,\phi) \quad = \quad -\mathcal{Y}_{j,j+\frac{1}{2}\delta,m}(\theta,\phi)$$

Even and odd excited states are

$$(a_1^+)^{2k}\Psi^{\epsilon}_{j,\delta,m}(r,\theta,\phi) = e_\epsilon \otimes \mathcal{Y}_{j,j-\frac{1}{2}\delta,m}(\theta,\phi)\cdot(-2)^k p^{\epsilon,\delta,\beta}_{2k,j}(r) r^{\epsilon\beta+\delta j+\frac{1}{2}\delta-1} e^{-\frac{1}{2}r^2},$$

$$(a_1^+)^{2k+1}\Psi^{\epsilon}_{j,\delta,m}(r,\theta,\phi) = i\sqrt{2} e_{-\epsilon} \otimes \mathcal{Y}_{j,j+\frac{1}{2}\delta,m}(\theta,\phi)\cdot(-2)^k p^{\epsilon,\delta,\beta}_{2k+1,j}(r) r^{\epsilon\beta+\delta j+\frac{1}{2}\delta-1} e^{-\frac{1}{2}r^2},$$

where $p^{\epsilon,\delta,\beta}_{2k,j}(r)$ and $p^{\epsilon,\delta,\beta}_{2k+1,j}(r)$ are r-dependent polynomials recursively determined by the Rodrigues-type formulas

$$p^{\epsilon,\delta,\beta}_{2k,j}(r) = \frac{1}{2^{2k}} \begin{pmatrix} r^{-\overline{\gamma}} e^{\frac{r^2}{2}} & 0 \end{pmatrix} \begin{pmatrix} 0 & \partial_r - r + \frac{\overline{\gamma}+2}{r} \\ \partial_r - r - \frac{\overline{\gamma}}{r} & 0 \end{pmatrix}^{2k} \begin{pmatrix} r^{\overline{\gamma}} e^{-\frac{r^2}{2}} \\ 0 \end{pmatrix},$$

$$p^{\epsilon,\delta,\beta}_{2k+1,j}(r) = \frac{1}{2^{2k+1}} \begin{pmatrix} r^{-\overline{\gamma}} e^{\frac{r^2}{2}} & 0 \end{pmatrix} \begin{pmatrix} 0 & \partial_r - r + \frac{\overline{\gamma}+2}{r} \\ \partial_r - r - \frac{\overline{\gamma}}{r} & 0 \end{pmatrix}^{2k+1} \begin{pmatrix} 0 \\ r^{\overline{\gamma}} e^{-\frac{r^2}{2}} \end{pmatrix},$$

where

$$\overline{\gamma} \equiv \gamma_{(j,\delta,\epsilon)}(\beta) = \epsilon\beta + \delta j + \tfrac{1}{2}\delta - 1.$$

It follows in particular, from $p^{\epsilon,\delta,\beta}_{0,j}(r) = 1$, that

$$p^{\epsilon,\delta,\beta}_{2,j}(r) = r^2 - \overline{\gamma} - \frac{3}{2}.$$

and so on.

The associated Laguerre polynomials $L_k^{(\gamma)}(x)$ are introduced through the position

$$L_k^{(\gamma)}(x) \quad = \quad \frac{x^{-\gamma} e^x}{k!} (\frac{d}{dx})^k x^{\gamma+k} e^{-x}.$$

They satisfy the identities

$$\begin{aligned} L_k^{(\gamma)}(x) \quad &= \quad L_k^{(\gamma+1)}(x) - L_{k-1}^{(\gamma+1)}(x), \\ x L_{k-1}^{(\gamma+1)}(x) \quad &= \quad (\gamma+k) L_{k-1}^{(\gamma)}(x) - k L_k^{(\gamma)}(x). \end{aligned}$$

Since

$$L_1^{(\gamma)}(x) \quad = \quad -x + \gamma - 1,$$

by setting

$$x = r^2, \qquad \gamma = \overline{\gamma} + \frac{1}{2},$$

we can identify

$$p_{2,j}^{\epsilon,\delta,\beta}(r) \quad = \quad -L_1^{(\overline{\gamma}+\frac{1}{2})}(r^2).$$

By assuming the Ansatz

$$p_{2k,j}^{\epsilon,\delta,\beta}(r) = C_k L_k^{(\overline{\gamma}+\frac{1}{2})}(r^2),$$

via induction one proves that

$$C_k = (-1)^k k!$$

The $p_{2k,j}^{\epsilon,\delta,\beta}(r)$ even and $p_{2k+1,j}^{\epsilon,\delta,\beta}(r)$ odd polynomials are expressed, in terms of the associated Laguerre polynomials, as

$$\begin{aligned} p_{2k,j}^{\epsilon,\delta,\beta}(r) &= (-1)^k k! L_k^{(\overline{\gamma}+\frac{1}{2})}(r^2), \\ p_{2k+1,j}^{\epsilon,\delta,\beta}(r) &= (-1)^{k+1} k! r L_k^{(\overline{\gamma}+\frac{3}{2})}(r^2). \end{aligned}$$

The normalizing factors are recovered from the orthogonal relations for the associated Laguerre polynomials, given by

$$\int_0^{+\infty} dx x^{\gamma} e^{-x} L_n^{(\gamma)}(x) L_m^{(\gamma)}(x) = \frac{\Gamma(n+\gamma+1)}{n!}\delta_{nm}.$$

Final results (orthonormal wave functions):

$$\Psi^{\epsilon}_{N,2k,j,\delta,m}(r,\theta,\phi) \quad = \quad e_{\epsilon} \otimes \mathcal{Y}_{j,j-\frac{1}{2}\delta,m}(\theta,\phi) \cdot M^{\overline{\gamma}}_{2k} L_k^{(\overline{\gamma}+\frac{1}{2})}(r^2) \cdot r^{\overline{\gamma}} e^{-\frac{r^2}{2}}$$

with

$$M^{\overline{\gamma}}_{2k} \quad = \quad \sqrt{\frac{(k!)\cdot 2}{\Gamma(k+\overline{\gamma}+\frac{3}{2})}}$$

and

$$\Psi^{\epsilon}_{N,2k+1,j,\delta,m}(r,\theta,\phi) \quad = \quad e_{-\epsilon} \otimes \mathcal{Y}_{j,j+\frac{1}{2}\delta,m}(\theta,\phi) \cdot M^{\overline{\gamma}}_{2k+1} L_k^{(\overline{\gamma}+\frac{3}{2})}(r^2) \cdot r^{\overline{\gamma}+1} e^{-\frac{r^2}{2}}$$

with

$$M^{\overline{\gamma}}_{2k+1} \quad = \quad \sqrt{\frac{(k!)\cdot 2}{\Gamma(k+\overline{\gamma}+\frac{5}{2})}}.$$

Dimensional reductions:

The $3D \rightarrow 2D$ case

Restrictions:

$$\partial\!\!\!/ = h_1\partial_1 + h_2\partial_2, \qquad r\!\!\!/ = x_1 h_1 + x_2 h_2, \qquad r = \sqrt{x_1^2 + x_2^2}$$

The $\overrightarrow{\mathbf{S}} \cdot \overrightarrow{\mathbf{L}}$ operator entering the Hamiltonians is now given by $S_3 L_3$ and is diagonal.

The resulting Hamiltonian $H_{2D,osc}$ corresponds to two copies of the two-dimensional 2×2 matrix Hamiltonians derived from the quantization of the $sl(2|1)$ worldline sigma-model with two propagating bosonic and two propagating fermionic fields:

$$H_{2D,osc} \quad = \quad -\frac{1}{2}(\partial_{x_1}^2 + \partial_{x_2}^2) \cdot \mathbb{I}_4 + \frac{1}{2r^2}(\beta^2 \mathbb{I}_4 + \beta N_F (1 + 2 \cdot \mathbb{I}_2 \otimes \sigma_3 L_3)) + \frac{1}{2} r^2 \mathbb{I}_4.$$

The $3D \rightarrow 1D$ case

Restrictions:

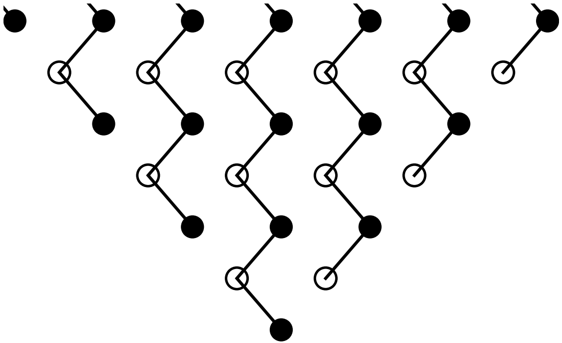
$$\not\partial = h_3\partial_3, \qquad \not r = x_3 h_3, \qquad r = \sqrt{x_3^2}.$$

The resulting $H_{1D,osc}$ deformed oscillator is (we set $x = x_3$)

$$H_{1D,osc} = -\frac{1}{2}\partial_x^2 \cdot \mathbb{I}_4 + \frac{1}{2x^2}(\beta^2 \cdot \mathbb{I}_4 + \beta N_F) + \frac{1}{2}x^2 \cdot \mathbb{I}_4,$$

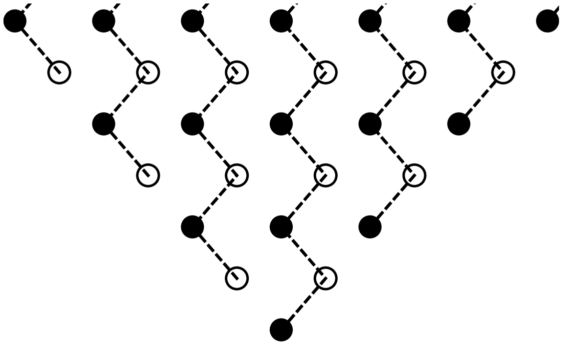
It coincides with the model derived from the quantization of the world-line sigma model induced by the $(1,4,3)$ supermultiplet.
The $H_{1D,osc}$ Hamiltonian possesses the larger $D(2,1;\alpha)$ spectrum-generating superalgebra, with $\alpha = \beta - \frac{1}{2}$.
The $sl(2|1) \subset D(2,1;\alpha)$ generators are sufficient to determine the ray vectors of the Hilbert space.
From the dimensional reduction viewpoint, the extra generators entering $D(2,1;\alpha)$ are associated with an emergent symmetry.

Original spectrum-generating superalgebra:



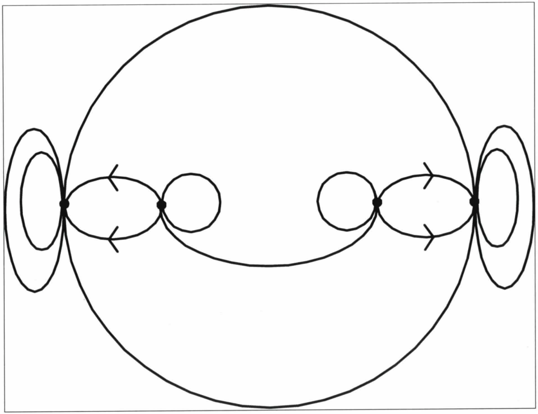
Superselected $2D$ oscillator. The bosonic (fermionic) eigenstates are represented by black (white) dots. The y axis labels the energy eigenvalues, the x axis labels the $so(2)$ spin components. The solid edges represent the action of the creation operator from the $osp(1|2) \subset sl(2|1)$ subalgebra. Infinite $osp(1|2)$ lwr's are required to produce the spectrum of the theory.

Mirrored spectrum-generating superalgebra:



A mirror dual: the dashed edges represent the action of the creation operator from the $osp(1|2)_C \subset sl(2|1)_C$ subalgebra, produced by a new set of "mirrored" operators. As before, infinite $osp(1|2)_C$ lwr's are required to produce the spectrum. On the other hand, any energy eigenstate can be obtained from the bosonic vacuum through a path combining both solid and dashed edges.

Thanks a lot for the attention!



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