QUANTUM-CLASSICAL DUALITY AND EMERGENT SPACE-TIME

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$$Tr\left[e^{\beta\sum_I H_I \hat{\Gamma}^I}\right] \overset{?}{\cong} \int \mathcal{D}x\ \rho(x)\ e^{\beta\sum_I H_I \Gamma^I(x)}$$

$$\overset{!}{=} \left[\int \mathcal{D}y\mathcal{D}x\ \delta(y) G'_{(x,T;y,0)} e^{\beta\sum_I H_I \Gamma^I(x)}\right]_{T=1}$$

based on arXiv:1903.06083

Motivations

- **Imaginary time formalism** (Felix Bloch and others)

$$\mathcal{Z}[\beta] = Tr\left[\exp\left(\beta\hat{H}\right)\right] \cong \int_{\varphi(0)=\varphi(\beta)} \mathcal{D}\varphi \;\; e^{\int_0^{\beta} d\tau L[\varphi(\tau)]}$$

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- **AdS/CFT correspondence** (Juan Maldacena and others)

$$\mathcal{Z}[J] = \langle\Omega|\exp\left(\int d^{D+1} J^i(x)\hat{O}_i(x)\right)|\Omega\rangle \cong \int_{\phi^i_{\partial M}=J^i} \mathcal{D}\phi \;\; e^{\int d^{D+2}x \mathcal{L}[\phi^i(x)]}$$

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Spinor operators

Consider N fermionic subsystems described by operators satisfying:

- Commutation relation if $a \neq b$

$$[\hat{\gamma}_a^j, \hat{\gamma}_b^k] = 0 \tag{1}$$

- Anti-commutation relation

$$\{\hat{\gamma}_a^j, \hat{\gamma}_a^k\} = 2\delta^{jk}\hat{I} \tag{2}$$

  where $a, b \in \{1, ..., N\}$ and $j, k \in \{1, ..., D\}$.

- Hermitian condition

$$\hat{\gamma}_a^j = \hat{\gamma}_a^{j\dagger}, \tag{3}$$

- Tracelessness condition

$$Tr\left[\left(\hat{\gamma}_{a_1}^{j_{1,1}} \ldots \hat{\gamma}_{a_1}^{j_{1,d_1}}\right)\left(\hat{\gamma}_{a_2}^{j_{2,1}} \ldots \hat{\gamma}_{a_2}^{j_{2,d_2}}\right) \ldots \left(\hat{\gamma}_{a_n}^{j_{n,1}} \ldots \hat{\gamma}_{a_n}^{j_{n,d_n}}\right)\right] = 0 \tag{4}$$

  where $1 \leq a_1 < \ldots < a_n \leq N$ and $1 \leq j_{k,1} < \ldots < j_{k,d_k} \leq D$ for all k.

EXAMPLES

- For $D = 2$ the spinor operators can be represented by tensor products of two out of three Pauli matrices (e.g. $\hat{X}$ and $\hat{Y}$),

$$
\begin{aligned}
\hat{\gamma}_1^1 &= \hat{X} \otimes \hat{I} \otimes \ldots \otimes \hat{I} \\
\hat{\gamma}_1^2 &= \hat{Y} \otimes \hat{I} \otimes \ldots \otimes \hat{I} \\
\hat{\gamma}_2^1 &= \hat{I} \otimes \hat{X} \otimes \ldots \otimes \hat{I} \\
\hat{\gamma}_2^2 &= \hat{I} \otimes \hat{Y} \otimes \ldots \otimes \hat{I} \\
&\ldots \\
\hat{\gamma}_N^1 &= \hat{I} \otimes \hat{I} \otimes \ldots \otimes \hat{X} \\
\hat{\gamma}_N^2 &= \hat{I} \otimes \hat{I} \otimes \ldots \otimes \hat{Y}.
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- For $D = 4$ the spinor operators. can be represented by tensor products of euclidean Dirac matrices
- Although the dimensionality D is kept arbitrary the two cases with $D = 1$ and $D = 3$ will turn out to be dual to simple classical models on S^0 and on S^2 configuration/target spaces.

HAMILTONIAN

- From the spinor operators we construct a Hamiltonian operator

$$\hat{H}_{\mathrm{q}} = \sum_{j_1 \ldots j_N \in \{0, \ldots, D\}} H_{j_1 \ldots j_N} \, \hat{\gamma}_1^{j_1} \ldots \hat{\gamma}_N^{j_N} \, . \tag{6}$$

where $\hat{\gamma}_a^0 \equiv \hat{I}$ and all of the components $H_{j_1 \ldots j_N}$ are real numbers.
- Quantum partition function can be expanded as power series

$$\mathcal{Z}_{\mathrm{q}}[H] = Tr\left[\exp\left(\beta \hat{H}_{\mathrm{q}}\right)\right] = \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \, Tr\left[\left(\sum_{j_1 \ldots j_N \in \{0, \ldots, D\}} H_{j_1 \ldots j_N} \, \hat{\gamma}_1^{j_1} \ldots \hat{\gamma}_N^{j_N}\right)^n\right]$$

- and each power of Hamiltonian operator into a *formal* sum

$$Tr\left[\left(\sum_{j_1 \ldots j_N \in \{0, \ldots, D\}} H_{j_1 \ldots j_N} \, \hat{\gamma}_1^{j_1} \ldots \hat{\gamma}_N^{j_N}\right)^n\right] = \sum_A h_A \, Tr\left[\hat{\Gamma}^A\right] \tag{7}$$

where h_A's represent products of $H_{j_1 \ldots j_N}$'s components and $\hat{\Gamma}^A$'s the corresponding products of the spinor operators.

COMBINATIONS OF TERMS

- Let $\sigma(A)$ be a set of all abstract-indices which are equivalent to A up to different combinations of terms from Hamiltonian.
- Then

$$\sum_A h_A \hat{\Gamma}^A = \sum_A \mu(A) h_A : \hat{\Gamma}^A : \tag{8}$$

where an ordered product of $\hat{\gamma}_a^j$ operators is given by

$$: \hat{\Gamma}^A := \theta(\hat{\Gamma}^A)\hat{\Gamma}^A$$

for some sign $\theta(\hat{\Gamma}^A) = \pm 1$ and the "average" sign is

$$\mu(A) = \frac{1}{|\sigma(A)|} \sum_{B \in \sigma(A)} \theta(\hat{\Gamma}^B) \tag{9}$$

- Then the trace of powers of Hamiltonian can be written in terms of ordered operators

$$Tr\left[\left(\sum_{j_1 \ldots j_N \in \{0,\ldots,D\}} H_{j_1 \ldots j_N} \hat{\gamma}_1^{j_1} \ldots \hat{\gamma}_N^{j_N}\right)^n\right] = \sum_A \mu(A) h_A Tr\left[: \hat{\Gamma}^A :\right].$$

COMBINATIONS OF TERMS

- $$Tr\left[\left(\sum_{j_1\dots j_N\in\{0,\dots,D\}} H_{j_1\dots j_N}\,\hat{\gamma}_1^{j_1}\dots\hat{\gamma}_N^{j_N}\right)^n\right] = \sum_A \mu(A) h_A Tr\left[:\hat{\Gamma}^A:\right]$$
- For example, if A represents $\left(H_{02}\hat{\gamma}_2^2\right)\left(H_{30}\hat{\gamma}_1^3\right)$, then

$$\begin{aligned} h_A &= H_{02}H_{30} \\ \hat{\Gamma}^A &= \hat{\gamma}_2^2\hat{\gamma}_1^3 \\ :\hat{\Gamma}^A: &= \hat{\gamma}_1^3\hat{\gamma}_2^2 \\ \theta(\hat{\Gamma}^A) &= 1 \\ \mu(A) &= (1+1)/2 = 1, \end{aligned}$$

but if A represents $\left(H_{03}\hat{\gamma}_2^3\right)\left(H_{02}\hat{\gamma}_2^2\right)$, then

$$\begin{aligned} h_A &= H_{03}H_{02} \\ \hat{\Gamma}^A &= \hat{\gamma}_2^3\hat{\gamma}_2^2 \\ :\hat{\Gamma}^A: &= \hat{\gamma}_2^2\hat{\gamma}_2^3 \\ \theta(\hat{\Gamma}^A) &= -1 \\ \mu(A) &= (1-1)/2 = 0. \end{aligned}$$

PARTITION FUNCTIONS

- Quantum partition function for N spinors with $D = 1$

$$\mathcal{Z}_{\mathrm{q}}[H] = \sum_{n=0}^{\infty} \frac{\beta^n}{n!} Tr \left[\left(\sum_{j_1 \ldots j_N \in \{0,1\}} H_{j_1 \ldots j_N} \hat{\gamma}_1^{j_1} \ldots \hat{\gamma}_N^{j_N} \right)^n \right] \tag{10}$$

- Classical partition function for N scalars x_a's,

$$\mathcal{Z}_{\mathrm{c}}[H] = \mathcal{N} \int \left(\prod_a dx_a \rho(x_a) \right) \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \left(\sum_{j_1 \ldots j_N \in \{0,1\}} H_{j_1 \ldots j_N} x_1^{j_1} \ldots x_N^{j_N} \right)^n \tag{11}$$

where $x_a^1 \equiv x_a$ and $x_a^0 \equiv 1$.

- The two systems are dual if

$$\mathcal{N} \int \left(\prod_a dx_a \rho(x_a) \right) \left(\sum_{j_1 \ldots j_N \in \{0,1\}} H_{j_1 \ldots j_N} x_1^{j_1} \ldots x_N^{j_N} \right)^n = Tr \left[\left(\sum_{j_1 \ldots j_N \in \{0,1\}} H_{j_1 \ldots j_N} \hat{\gamma}_1^{j_1} \ldots \hat{\gamma}_N^{j_N} \right)^n \right]$$

Partition functions

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or using the abstract-indices notation

$$\mathcal{N} \int \left(\prod_a dx_a \rho(x_a) \right) \sum_A h_A \, X^A = \sum_A \mu(A) h_A Tr \left[: \hat{\Gamma}^A : \right] . \tag{12}$$

where X^A is the corresponding products of scalars.

MEASURE OF INTEGRATION

- Since all operators $\hat{\gamma}_a^1$'s commute their products are such that $\hat{\Gamma}^A =: \hat{\Gamma}^A :$ and, thus, $\mu(A) = 1$ for all A.
- Then by matching individual terms we get

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- The ordered product of operators $: \hat{\Gamma}^A :$ either contains
  - even number of $\hat{\gamma}_a^1$ operators for every a

$$\Rightarrow Tr \left[\hat{\Gamma}^A \right] = Tr \left[\hat{I} \right] \equiv \mathcal{N} \tag{14}$$

  - or $\exists$ at least one a for which there is an odd number of $\hat{\gamma}_a^1$'s

$$\Rightarrow Tr \left[\hat{\Gamma}^A \right] = 0 \tag{15}$$

- Then the measure of integration $\rho(x_a)$ should be such that all odd statistical moments vanish and all even statistical moment are the same,

$$\int (x_a)^n \rho(x_a) dx_a = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{16}$$

CLASSICAL DUAL

- But this is can be easily achieved with

$$\rho(x_a) = \frac{\delta(x_a - 1) + \delta(x_a + 1)}{2} \tag{17}$$

  which corresponds to a classical partition function

$$\mathcal{Z}_c[H] = \mathcal{N} \sum_{x_1 \dots x_N \in \{1,-1\}} \exp\left(\beta \sum_{j_1 \dots j_N \in \{0,1\}} H_{j_1 \dots j_N} \, x_1^{j_1} \dots x_N^{j_N} \right). \tag{18}$$

- We conclude that the quantum system is dual to a classical system

$$\hat{H}_q = \sum_{j_1 \dots j_N \in \{0,1\}} H_{j_1 \dots j_N} \, \hat{\gamma}_1^{j_1} \dots \hat{\gamma}_N^{j_N} \quad \Leftrightarrow \quad H_c = \sum_{j_1 \dots j_N \in \{0,1\}} H_{j_1 \dots j_N} \, x_1^{j_1} \dots x_N^{j_N}$$

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where x_a are the classical spinors (or classical scalars on S^0 target space)
- Note that the eigenvalues of the quantum Hamiltonian must be

$$E_x = \sum_{j_1 \dots j_N \in \{0,1\}} H_{j_1 \dots j_N} \, x_1^{j_1} \dots x_N^{j_N}, \tag{19}$$

where $x \in \{-1, 1\}^N$.

PARTITION FUNCTIONS

- Quantum partition function for a single spinor ($N = 1$), but with $D > 1$

$$\mathcal{Z}_{\mathrm{q}}[H] = \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \, Tr \left[\left(\sum_{j \in \{1,\dots,D\}} H_j \, \hat{\gamma}^j \right)^n \right] \tag{20}$$

- Classical partition function for a system of D scalars

$$\mathcal{Z}_{\mathrm{c}}[H] = \mathcal{N} \int d^D x \rho(x) \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \left(\sum_{j \in \{1,\dots,D\}} H_j \, x^j \right)^n . \tag{21}$$

- The two systems are dual if

$$\begin{aligned} \mathcal{N} \int d^D x \rho(x) \sum_A h_A \, X^A &= \sum_A \mu(A) h_A Tr \left[: \hat{\Gamma}^A : \right] , \\ \mathcal{N} \int d^D x \rho(x) X^A &= \mu(A) Tr \left[: \hat{\Gamma}^A : \right] \\ \mathcal{N} \int d^D x \rho(x) \prod_k (x^k)^{2n_k} &= \mu(A) Tr \left[\prod_k (\hat{\gamma}^k)^{2n_k} \right] \\ \int d^D x \rho(x) \prod_k (x^k)^{2n_k} &= \mu(A) \end{aligned} \tag{22}$$

Multinomials

- Consider the following two multinomials:
  - a sum of commuting scalars raised to some even power

$$\left(x^1 + x^2 + \ldots + x^D\right)^{2K} = \sum_{m_1+\ldots+m_D=2K} \frac{(m_1 + \ldots + m_D)!}{(m_1)!\ldots(m_D)!}(x^1)^{m_1}\ldots(x^D)^{m_D}$$

$$= \sum_{m_1+\ldots+m_D=2K} \boxed{\frac{(\sum_k m_k)!}{\prod_k (m_k)!}}(x^1)^{m_1}\ldots(x^D)^{m_D}$$

  - a sum of anti-commuting operators raised to the same power

$$\left(\hat{\gamma}^1 + \hat{\gamma}^2 + \ldots + \hat{\gamma}^D\right)^{2K} = \left((\hat{\gamma}^1)^2 + (\hat{\gamma}^2)^2 + \ldots + (\hat{\gamma}^D)^2\right)^K$$

$$= \sum_{n_1+\ldots+n_D=K} \frac{(n_1 + \ldots + n_D)!}{(n_1)!\ldots(n_D)!}(\hat{\gamma}^1)^{2n_1}\ldots(\hat{\gamma}^D)^{2n_D}.$$

$$= \sum_{n_1+\ldots+n_D=K} \boxed{\frac{(\sum_k n_k)!}{\prod_k (n_k)!}}(\hat{\gamma}^1)^{2n_1}\ldots(\hat{\gamma}^D)^{2n_D}.$$

- Separate terms in the expansion of operators represent products of $\hat{\gamma}^k$'s applied in different orders (or combinations $\sigma(A)$) and we are interested in products of $m_1 = 2n_1$ of $\hat{\gamma}^1$'s, $m_2 = 2n_2$ of $\hat{\gamma}^2$'s, etc.

MEASURE OF INTEGRATION

- $$\mu(A) = \frac{1}{|\sigma(A)|} \sum_{B \in \sigma(A)} \theta(\hat{\Gamma}^B) = \frac{\prod_k (2n_k)!}{(\sum_k 2n_k)!} \frac{(\sum_k n_k)!}{\prod_k n_k!} = \int d^D x \, \rho(x) \prod_k (x^k)^{2n_k}$$

  where A can represent an arbitrary product of terms with $2n_k$ of $\hat{\gamma}^k$'s
- The moments generating function of $\rho(x)$

$$\begin{aligned} M(p_1, ..., p_D) &= \sum_{n_1,\dots,n_D} \left(\frac{\prod_k (2n_k)!}{(\sum_k 2n_k)!} \frac{(\sum_k n_k)!}{\prod_k n_k!} \right) \frac{p_1^{2n_1} ... p_D^{2n_D}}{\prod_k (2n_k)!} = \\ &= \cosh\left(\sqrt{p_1^2 + ... + p_D^2} \right) \end{aligned} \tag{23}$$

- The corresponding characteristic function is

$$M(ip_1, ..., ip_D) = \cos\left(\sqrt{p_1^2 + ... + p_D^2} \right) = \cos\left(\sqrt{\sum_k p_k^2} \right) \tag{24}$$

  whose inverse Fourier transform is the desired measure of integration

$$\rho(x) = \int \frac{d^D p}{(2\pi)^D} \cos\left(\sqrt{\sum_k p_k^2} \right) \exp\left(i \sum_k x^k p_k \right). \tag{25}$$

EMERGENT SPACE-TIME

- For $D = 1$

$$\rho(x) = \int \frac{dp}{2\pi} \cos\left(\sqrt{p^2}\right) \exp(ixp) = \frac{1}{2}\left(\delta(x+1) + \delta(x-1)\right), \quad (26)$$

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- For arbitrary D we note that

$$\varphi(x^\mu) = \varphi(\vec{x}, x^0) \equiv \int \frac{d^D p}{(2\pi)^D} \cos\left(x^0 \sqrt{\sum_k (p_k)^2}\right) \exp\left(i \sum_k p_k x^k\right) \tag{27}$$

solves a D+1-dimensional wave equation,

$$\left((\partial_0)^2 - \sum_k (\partial_k)^2\right) \varphi(x^\mu) = 0, \tag{28}$$

with initial conditions

$$\varphi(\vec{x}, 0) = \delta^{(D)}(\vec{x}) \tag{29}$$

$$\partial_0 \varphi(\vec{x}, 0) = 0 \tag{30}$$

EXTENDED PARTITION FUNCTION

- Solution of the D+1-dimensional wave equation is given by

$$\varphi(x^\mu) = \int d^D y\, \partial_0 G(\vec{x}, x^0; \vec{y}, 0)\delta^{(D)}(\vec{y}) = \partial_0 G(\vec{x}, x^0) \tag{31}$$

where (with a slight abuse of notations)

$$G(x^\mu; y^\mu) = G(\vec{x} - \vec{y}, x^0 - y^0) = G(x^\mu - y^\mu) = G(\vec{x} - \vec{y}, x^0 - y^0)$$

is the retarded Green's function of D+1-dim. d'Alembert operator.

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is the retarded Green's function of D+1-dim. d'Alembert operator.

- Extended (into "temporal" direction T) partition function is defined as

$$\begin{aligned} \mathcal{Z}_c[H, T] &= \mathcal{N} \int d^D x\, \varphi(\vec{x}, T) \exp\left(\beta \sum_{j\in\{1,\dots,D\}} H_j\, x^j\right) \\ &= \mathcal{N} \int d^D y \int d^D x \exp\left(\beta \sum_{j\in\{1,\dots,D\}} H_j\, x^j\right) \partial_0 G(\vec{x}, T; \vec{y}, 0)\delta^{(D)}(\vec{y}) \end{aligned} \tag{32}$$

- By construction it satisfies the desired duality condition

$$\mathcal{Z}_c[H, 1] = \mathcal{Z}_q[H] \tag{33}$$

and also normalization conditions

$$\mathcal{Z}_c[H, 0] = \mathcal{N}. \tag{34}$$

Existence of duality

- Quantum partition function for $N > 1$ quantum spinors with $D > 1$

$$\mathcal{Z}_{\mathrm{q}}[H] = \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \, Tr\left[\left(\sum_{j_1 \dots j_N \in \{0,\dots,D\}} H_{j_1 \dots j_N} \, \hat{\gamma}_1^{j_1} \dots \hat{\gamma}_N^{j_N}\right)^n\right]$$

- Classical partition function for a system of ND classical scalars

$$\mathcal{Z}_{\mathrm{c}}[H] = \mathcal{N} \int \left(\prod_a d^D x_a\right) \rho(x) \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \left(\sum_{j_1 \dots j_N \in \{0,\dots,D\}} H_{j_1 \dots j_N} \, x_1^{j_1} \dots x_N^{j_N}\right)^n$$

- The two systems are dual if all odd moments vanish and

$$\cancel{\mathcal{N}} \int \left(\prod_a d^D x_a\right) \rho(x) X^A = \cancel{\mathcal{N}} \int \left(\prod_a d^D x_a\right) \rho(x) \prod_{a,k} \left(x_a^k\right)^{2n_k^a} = \cancel{Tr\left[:\hat{\Gamma}^A:\right]} \mu(A)$$

- Then the measure only exists if

$$:\hat{\Gamma}^A :=: \hat{\Gamma}^B : \quad \Rightarrow \quad \mu(A) = \mu(B). \tag{36}$$

i.e. even if A and B are not in the same combination class, $\sigma(A) \neq \sigma(B)$, but the corresponding products of operators are the same, $:\hat{\Gamma}^A :=: \hat{\Gamma}^B :$, the statistical moments must also be the same, $\mu(A) = \mu(B)$.

Separable measure

- Consider a Hamiltonian with components which can be expressed as

$$H_{j_1\ldots j_N} = \sum_{k_1\ldots k_N \in \{0,1\}} \mathcal{H}_{k_1\ldots k_N} \eta_{1,j_1}^{k_1} \ldots \eta_{N,j_N}^{k_N} \tag{37}$$

where we assume that $H_{0\ldots 0} = 0$ and

$$\eta_{a,j}^{0} = \delta_{0j} \tag{38}$$

$$\eta_{a,0}^{k} = \delta_{k0} \tag{39}$$

- Then the Hamiltonian operator

$$\hat{H}_{\mathrm{q}} = \sum_{j_1\ldots j_N \in \{0,\ldots,D\}} H_{j_1\ldots j_N} \hat{\gamma}_1^{j_1} \ldots \hat{\gamma}_N^{j_N} \tag{40}$$

$$= \sum_{k_1\ldots k_N \in \{0,1\}} \mathcal{H}_{k_1\ldots k_N} \hat{\eta}_1^{k_1} \ldots \hat{\eta}_N^{k_N}$$

where

$$\hat{\eta}_a = \hat{\eta}_a^1 = \sum_{j\in\{0,\ldots,D\}} \eta_{a,j}^1 \hat{\gamma}_a^j = \sum_{j\in\{1,\ldots,D\}} \eta_{a,j}^1 \hat{\gamma}_a^j \tag{41}$$

and

$$\hat{\eta}_a^0 = \sum_{j\in\{0,\ldots,D\}} \eta_{a,j}^0 \hat{\gamma}_a^j = \sum_{j\in\{0,\ldots,D\}} \delta_{0j} \hat{\gamma}_a^j = \hat{\gamma}_a^0 = \hat{I}. \tag{42}$$

Dual system

- Since the combined operators satisfy a commutation relation

$$[\hat{\eta}_a, \hat{\eta}_b] = 0 \tag{43}$$

we can essentially follow the above analysis with

$$\begin{aligned}\mathcal{Z}_c[H] &= \mathcal{N} \int \left(\prod_a d^D x_a \rho(x_a) \right) \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \left(\sum_{j_1 \dots j_N \in \{0,\dots,D\}} H_{j_1\dots j_N} x_1^{j_1} \dots x_N^{j_N} \right)^n \\ &= \mathcal{N} \int \left(\prod_a d^D x_a \rho(x_a) \right) \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \left(\sum_{k_1 \dots k_N \in \{0,1\}} \mathcal{H}_{k_1\dots k_N} \chi_1^{k_1} \dots \chi_N^{k_N} \right)^n\end{aligned}$$

where

$$\chi_a = \chi_a^1 = \sum_{j\in\{1,\dots,D\}} \eta^1_{a,j} x_a^j \qquad \chi_a^0 = 1$$

- Result:

$$\mathcal{Z}_c[H,T] = \mathcal{N} \int \left(\prod_a d^D x_a \, \partial_0 G(\vec{x}_a, T_a) \right) \exp\left(\beta \sum_{j_1 \dots j_N \in \{0,1\}} H_{j_1\dots j_N} x_1^{j_1} \dots x_N^{j_N} \right),$$

EXAMPLES

- Note that the measure of integration is already normalized,

$$\int \left(\prod_a d^D x_a \, \partial_0 G(\vec{x}_a, T_a) \right) = 1, \tag{44}$$

  but it can be interpreted as probability only if $\partial_0 G(\vec{x}_a, T_a) \geq 0$.
- For example, when $D = 1$

$$\mathcal{Z}_c[H, T] = \mathcal{N} \int \prod_a \left(\frac{dx_a}{2} \left(\delta(x_a^1 - T_a) + \delta(x_a^1 + T_a) \right) \right) \exp\left(\beta H_c\right), \tag{45}$$

  in agreement with pervious results, or when $D = 3$

$$\mathcal{Z}_c[H, T] = \mathcal{N} \int \prod_a \left(\frac{d^3 x_a}{4\pi T_a^2} \delta \left(\sum_k \left(x_a^k\right)^2 - T_a^2 \right) \right) \exp\left(\beta H_c\right). \tag{46}$$

- Of course there is no reason to expect that the measure will remain positive for more general quantum systems and then the dual system defined in a similar manner would not be classical *per se*.

Conclusion

- We considered a quantum-classical duality mapping between quantum and classical systems of the form

$$\mathcal{Z}[H] = Tr\left[\exp\left(\beta \sum_I H_I \hat{\Gamma}^I\right)\right] \quad \cong \quad \int \mathcal{D}x\ \rho(x)\ e^{\beta \sum_I H_I \Gamma^I(x)}$$

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- For a system of quantum spinors the dual classical system consists of scalars with only linear functions $\Gamma^I(x)$'s, but non-trivial measures of integrations $\rho(x)$.
- Measure is given by relativistic Green's functions which suggest a possible mechanism for emergence of a classical space-time from anti-commutativity of quantum operators or vice versa