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Equivalence groupoids, normalization property and group classification problems for evolution equations

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It is widely known that there is no general theory for integration of nonlinear partial differential equations (PDEs). Nevertheless, many special cases of complete integration or finding particular solutions are related to appropriate changes of variables.

Nondegenerate point transformations that leave a differential equation invariant and form a connected Lie group are called Lie symmetries of this equation. Transformations of this kind are ones which are mostly used.

This places the transformation methods among the most powerful analytic tools currently available in the study of nonlinear PDEs.

The presence of nontrivial symmetry properties is one of the distinctive features which differ equations describing natural phenomena from other possible ones

All the basic equations of mathematical physics, i.e. the equations of Newton, Laplace, d'Alembert, Euler-Lagrange, Lame, Hamilton-Jacobi, Maxwell, Schrödinger etc., have nontrivial symmetry properties.
It means that manifolds of their solutions are invariant with respect to multi-parameter group of continuous transformations (Lie group of transformations) with large number of parameters.

The requirement of invariance of an equation under a group enables us in some cases to select this equation from a wide set of other admissible ones.

For example, there is the only one system of Poincaré-invariant first-order partial differential equations of for two real vectors E and H, and this is the system of Maxwell equations.

Group classification problem of differential equations consists in the specification of non-equivalent cases of such equations which possess the extensions of Lie symmetries.

The systematic study of transformational properties of classes of nonlinear PDEs was initiated in 1991 by Kingston and Sophocleous. These authors later named the transformations related two particular equations in a class of PDEs form-preserving transformations, because such transformations preserve the form of the equation in a class and change only its arbitrary elements.

Only a year later in 1992 Gazeau and Winternitz started to investigate such transformations in classes of PDEs calling them allowed transformations.

Rigorous definitions and developed theory on the subject was proposed later by Popovych.
As formalization of notion of form-preserving (allowed) transformations he suggested the term admissible transformation.
In brief, an admissible transformation is a triple consisting of two fixed equations from a class and a transformation that links these equations.
The set of admissible transformations considered with the standard operation of composition of transformations is also called the equivalence groupoid.

Equivalence group

There exist several kinds of equivalence groups depending on restrictions that are imposed on the transformations. The usual equivalence group of the class $\mathcal{L}|_{\mathcal{S}}$ consists of the nondegenerate point transformations in the space of variables and arbitrary elements, which preserve the whole class $\mathcal{L}|_{\mathcal{S}}$ and are projectable on the variable space, i.e., the transformation components corresponding to independent and dependent variables do not depend on arbitrary elements.

Generalized extended equivalence group

Restrictions on transformations can be weakened in two directions. We admit the transformations of the variables t, x and u can depend on arbitrary elements
(the prefix "generalized" [Meleshko, 1994]).

The explicit form of the new arbitrary elements $(\tilde f, \tilde g, \tilde h, \tilde n, \tilde m)$ is determined via (t, x, u, f, g, h, n, m) in some non-fixed (possibly, nonlocal) way
(the prefix "extended" [Ivanova&Popovych&Sophocleous, 2005]).

Normalization property

The class $\mathcal{L}|_{\mathcal{S}}$ is called normalized in the usual (resp. generalized, resp. extended, resp. generalized extended) sense if the equivalence groupoid of this class is induced by transformations from its equivalence group of the corresponding type.

Classes which are most convenient for investigation are those normalized in the usual sense.

Class normalized in any sense is always better than one that is not normalized.

Consider the general class of second-order evolution equations,

$$u_t = H(t, x, u, u_x, u_{xx}), \quad H_{u_{xx}} \neq 0.$$

Any point transformation $\mathcal{T}$ relating two fixed equations $u_t = H$ and $\tilde{u}_{\tilde{t}} = \tilde{H}$ from this class has the form
$\tilde{t} = T(t)$, $\tilde{x} = X(t, x, u)$, $\tilde{u} = U(t, x, u)$ with $T_t(X_x U_u - X_u U_x) \neq 0$.

$$\tilde{u}_{\tilde{t}} = \frac{D_t U D_x X - D_x U D_t X}{T_t D_x X}, \quad \tilde{u}_{\tilde{x}} = \frac{D_x U}{D_x X}, \quad \tilde{u}_{\tilde{x}\tilde{x}} = \frac{1}{D_x X} D_x \left(\frac{D_x U}{D_x X} \right),$$

where $D_t = \partial_t + u_t \partial_u + u_{tt} \partial_{u_t} + u_{tx} \partial_{u_x} + \dots$ and
$D_x = \partial_x + u_x \partial_u + u_{tx} \partial_{u_t} + u_{xx} \partial_{u_x} + \dots$

Proposition 1.

The class $u_t = H(t, x, u, u_x, u_{xx}), \quad H_{u_{xx}} \neq 0$, is normalized in the usual sense. Its equivalence group is formed by the transformations

$$\tilde{t} = T(t), \quad \tilde{x} = X(t, x, u), \quad \tilde{u} = U(t, x, u), \quad T_t(X_x U_u - X_u U_x) \neq 0,$$
$$\tilde{H} = \frac{X_x U_u - X_u U_x}{T_t D_x X} H + \frac{U_t D_x X - X_t D_x U}{T_t D_x X}.$$

The subclass of the above class singled out by the condition $H_{u_{xx} u_{xx}} = 0$ has the same equivalence transformation components for variables.

Proposition 2.

The class of quasilinear second-order evolution equations,

$$u_t = G(t,x,u,u_x)u_{xx} + F(t,x,u,u_x), \quad G \neq 0,$$

is normalized in the usual sense. Its equivalence group is formed by the transformations

$$\tilde t = T(t), \quad \tilde x = X(t,x,u), \quad \tilde u = U(t,x,u), \quad T_t(X_xU_u - X_uU_x) \neq 0,$$

$$\tilde G = \frac{(D_xX)^2}{T_t}G, \quad \tilde F = \frac{X_xU_u - X_uU_x}{T_tD_xX}F + \frac{U_tD_xX - X_tD_xU}{T_tD_xX} +$$

$$\frac{(X_{xx} + 2X_{xu}u_x + X_{uu}u_x^2)D_xU - (U_{xx} + 2U_{xu}u_x + U_{uu}u_x^2)D_xX}{T_tD_xX}G.$$

Proposition 3.

The class

$$u_t = G(t,x,u)u_{xx} + F(t,x,u,u_x), \quad G \neq 0,$$

is normalized in the usual sense. Its equivalence group comprises the transformations

$$\tilde t = T(t), \quad \tilde x = X(t,x), \quad \tilde u = U(t,x,u), \quad \tilde G = \frac{X_x^2}{T_t}G,$$

$$\tilde F = \frac{U_u}{T_t}F + \frac{U_tX_x - X_tD_xU}{T_tX_x} + \frac{X_{xx}D_xU - (U_{xx} + 2U_{xu}u_x + U_{uu}u_x^2)X_x}{T_tX_x}G,$$

where $T_tX_xU_u \neq 0$.

Proposition 4.

The class $u_t = G(t,x,u)u_{xx} + \sum_{k=0}^{n} F^k(t,x,u)u_x^k,\ n \geq 2, \quad G \neq 0$, is normalized in the usual sense. Its equivalence group consists of the transformations $\tilde{t} = T(t)$, $\tilde{x} = X(t,x)$, $\tilde{u} = U(t,x,u)$, $\tilde{G} = \dfrac{X_x^2}{T_t} G$ and the transformation components for the arbitrary elements F^k, $k = 0, \dots, n$, are found as solutions of the algebraic system resulting from the splitting w.r.t. different powers of u_x of the equation

$$\sum_{k=0}^{n} \tilde{F}^k \left(\frac{U_u}{X_x} u_x + \frac{U_x}{X_x} \right)^k = \frac{1}{T_t X_x} \Big[X_x U_u \sum_{k=0}^{n} F^k u_x^k + U_t X_x - X_t D_x U + \left(X_{xx} D_x U - (U_{xx} + 2U_{xu} u_x + U_{uu} u_x^2) X_x \right) G \Big].$$

Proposition 5.

The class
$u_t = G(t,x,u)u_{xx} + F^2(t,x,u)u_x^2 + F^1(t,x,u)u_x + F^0(t,x,u), \quad G \neq 0,$
is normalized in the usual sense. Its equivalence group consists of the transformations $\tilde t = T(t)$, $\tilde x = X(t,x)$, $\tilde u = U(t,x,u)$, $\tilde G = \dfrac{X_x^2}{T_t}G$,

$\tilde F^2 = \dfrac{X_x^2}{T_t U_u^2}\left(U_u F^2 - U_{uu}G\right)$, $\tilde F^1 =$
$\dfrac{1}{T_t U_u}\left(2\dfrac{X_x U_x}{U_u}(U_{uu}G - U_u F^2) + X_x U_u F^1 - X_t U_u + (X_{xx}U_u - 2U_{xu}X_x)G\right),$
$\tilde F^0 = \dfrac{1}{T_t}\left[\dfrac{U_x^2}{U_u}F^2 - U_x F^1 + U_u F^0 + U_t + \left(2\dfrac{U_x}{U_u}U_{xu} - U_{xx} - \dfrac{U_x^2}{U_u^2}U_{uu}\right)G\right],$
where $T_t X_x U_u \neq 0$.

Proposition 6.

The class

$$u_t = G(t,x,u)u_{xx} + F^1(t,x,u)u_x + F^0(t,x,u), \quad G \neq 0,$$

is normalized in the usual sense. Its equivalence group comprises the transformations

$$\tilde t = T(t), \quad \tilde x = X(t,x), \quad \tilde u = U^1(t,x)u + U^0(t,x), \quad T_tX_xU^1 \neq 0,$$

$$\tilde G = \frac{X_x^2}{T_t}G, \quad \tilde F^1 = \frac{1}{T_tU^1}\left(X_xU^1F^1 - X_tU^1 + (X_{xx}U^1 - 2U_x^1X_x)G\right),$$

$$\tilde F^0 = \frac{1}{T_t}\Big[U^1F^0 - (U_x^1u + U_x^0)F^1 + U_t^1u + U_t^0 + \\ + \left(2\frac{U_x^1}{U^1}(U_x^1u + U_x^0) - U_{xx}^1u - U_{xx}^0\right)G\Big].$$

Consider one more subclass of the class

$$u_t = G(t,x,u)u_{xx} + F^2(t,x,u)u_x^2 + F^1(t,x,u)u_x + F^0(t,x,u), \quad (1)$$

for which the condition $U_{uu} = 0$ holds for admissible transformations. This is the subclass singled out by the condition $F^2 = G_u$,

$$u_t = (G(t,x,u)u_x)_x + K(t,x,u)u_x + P(t,x,u), \quad G \neq 0.$$

This class can be written in the form
$u_t = Gu_{xx} + G_u u_x^2 + (G_x + K)u_x + P$, where connections between arbitrary elements of the latter class and class (1) are given by the formulas $F^2 = G_u$, $F^1 = G_x + K$, $F^0 = P$.

Proposition 7.

The class $u_t=(G(t,x,u)u_x)_x+K(t,x,u)u_x+P(t,x,u)$ is normalized in the usual sense. Its equivalence group is formed by the transformations

$$\tilde t=T(t),\quad \tilde x=X(t,x),\quad \tilde u=U^1(t,x)u+U^0(t,x),\quad T_tX_xU^1\neq 0,$$

$$\tilde G=\frac{X_x^2}{T_t}G,\quad \tilde K=\frac{X_x}{T_t}\left[K-\left(\frac{X_{xx}}{X_x}+2\frac{U^1_x}{U^1}\right)G-2(U^1_xu+U^0_x)\frac{G_u}{U^1}-\frac{X_t}{X_x}\right],$$

$$\tilde P=\frac{1}{T_t}\left[U^1P+\frac{(U^1_xu+U^0_x)^2}{U^1}G_u-(U^1_xu+U^0_x)(G_x+K)+U^1_tu+U^0_t+\right.$$
$$\left.\left(2\frac{U^1_x}{U^1}(U^1_xu+U^0_x)-U^1_{xx}u-U^0_{xx}\right)G\right].$$

The class

$$u_t = (G(t,x,u)u_x)_x + P(t,x,u), \quad G \neq 0, \tag{2}$$

is not normalized anymore.
It's equivalence group is comprised of the transformations

$$\tilde t = T(t), \quad \tilde x = \delta_1 x + \delta_2, \quad \tilde u = U^1(t)u + U^0(t), \quad T_t U^1 \delta_1 \neq 0,$$
$$\tilde G = \frac{\delta_1^2}{T_t} G, \quad \tilde P = \frac{1}{T_t}\left(U^1 P + U^1_t u + U^0_t\right).$$

If G does not satisfy the equation of the form

$$(au+b)G_u + c\,G + d = 0,$$

then class (2) is normalized.

We also consider the class

$$S(t,x)u_t=(G(t,x,u)u_x)_x+K(t,x,u)u_x+P(t,x,u),\quad SG\neq 0.$$

In particular, the classes of variable-coefficient diffusion–reaction equations $f(x)u_t=(g(x)A(u)u_x)_x+h(x)B(u)$ and diffusion–convection equations $f(x)u_t=(g(x)A(u)u_x)_x+h(x)B(u)u_x$ are subclasses of this class.
The coefficient $S(t,x)$ can be gauged to one by the family of point transformation

$$\tilde t=t,\quad \tilde x=\int_{x_0}^{x}S(t,y)\,\mathrm{d}y,\quad \tilde u=u.$$

Nevertheless, we will consider this class separately since its transformational properties become more complicated.

Proposition 8. Any point transformation between two equations from the class

$$S(t,x)u_t = (G(t,x,u)u_x)_x + K(t,x,u)u_x + P(t,x,u)$$

has the form

$$\tilde t = T(t),\quad \tilde x = X(t,x),\quad \tilde u = U^1(t,x)u + U^0(t,x),\quad T_tX_xU^1\neq 0.$$

Then arbitrary elements are related via the formulas $\dfrac{\tilde G}{\tilde S} = \dfrac{X_x^2}{T_t}\frac{G}{S}$,

$$\frac{\tilde K + \tilde G_{\tilde x}}{\tilde S} = \frac{X_x}{T_tS}\left[K + G_x + \left(\frac{X_{xx}}{X_x} - 2\frac{U^1_x}{U^1}\right)G - 2(U^1_xu + U^0_x)\frac{G_u}{U^1} - \frac{X_t}{X_x}S\right],$$

$$\frac{\tilde P}{\tilde S} = \frac{1}{T_tS}\left[U^1P + \frac{(U^1_xu + U^0_x)^2}{U^1}G_u - (U^1_xu + U^0_x)(K + G_x) + (U^1_tu + U^0_t)S + \left(2\frac{U^1_x}{U^1}(U^1_xu + U^0_x) - U^1_{xx}u - U^0_{xx}\right)G\right].$$

It is obvious that transformation properties of the class

$$S(t,x)u_t = (G(t,x,u)u_x)_x + K(t,x,u)u_x + P(t,x,u), \quad SG \neq 0.$$

become more complicated in comparison with those of its subclass with $S = 1$. Transformations are defined only for fractions of arbitrary elements. It is explained by the fact that this class admits peculiar gauge equivalence transformation (an equivalence transformation for which independent and dependent variables do not transform but only arbitrary elements). This is the transformation

$$\tilde{S} = Z(t,x,S), \quad \tilde{G} = \frac{G}{S}Z, \quad \tilde{K} = \frac{K}{S}Z - G\left(\frac{Z}{S}\right)_x, \quad \tilde{P} = \frac{P}{S}Z,$$

where Z is an arbitrary smooth function of its variables with $Z_S \neq 0$.

Proposition 9.
The class $S(t,x)u_t = (G(t,x,u)u_x)_x + K(t,x,u)u_x + P(t,x,u)$ is normalized. It's equivalence group comprises the transformations

$$\tilde t = T(t),\quad \tilde x = X(t,x),\quad \tilde u = U^1(t,x)u + U^0(t,x),\quad T_tX_xU^1 \neq 0,$$

$$\tilde S = Z(t,x,S),\quad \tilde G = \frac{X_x^2}{T_t}\frac{G}{S}Z,$$

$$\tilde K = \frac{X_xZ}{T_tS}\left[K - \left(\frac{X_{xx}}{X_x} + 2\frac{U^1_x}{U^1}\right)G - 2(U^1_xu + U^0_x)\frac{G_u}{U^1} - \frac{X_t}{X_x}S\right] - \frac{X_x}{T_t}G\left(\frac{Z}{S}\right)_x,$$

$$\tilde P = \frac{Z}{T_tS}\Bigg[U^1P + \frac{(U^1_xu + U^0_x)^2}{U^1}G_u - (U^1_xu + U^0_x)(K + G_x) + (U^1_tu + U^0_t)S +$$
$$\left(2\frac{U^1_x}{U^1}(U^1_xu + U^0_x) - U^1_{xx}u - U^0_{xx}\right)G\Bigg].$$

Vaneeva O.O. et al. (2014) Physica Scripta, 89 (3) 038003.

The class

$$u_t = F(t)u_n + G(t, x, u_0, u_1, \ldots, u_{n-1}), \quad F \neq 0, \quad G_{u_i u_{n-1}} = 0,$$

where $i = 1, \ldots, n-1, \quad n \geqslant 2$, is normalized.
Its usual equivalence group consists of the transformations

$$\tilde{t} = T(t), \quad \tilde{x} = X^1(t)x + X^0(t), \; \tilde{u} = U^1(t,x)u + U^0(t,x), \; \tilde{F} = \frac{(X^1)^n}{T_t}F,$$

$$\tilde{G} = \frac{U^1}{T_t}G - \left(\sum_{k=0}^{n-1}\binom{n}{k}U^1_{n-k}u_k + U^0_n\right)\frac{F}{T_t} + \frac{U^1_t}{T_t}u +$$

$$+\frac{U^0_t}{T_t} - \frac{X^1_t x + X^0_t}{T_t X^1}(U^1 u_x + U^1_x u + U^0_x),$$

where $T_t X^1 U^1 \neq 0$.

We study generalized Kawahara equations

$$u_t + \alpha(t)f(u)u_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0, \qquad (3)$$

from the Lie symmetry point of view. Here f, α, β and σ are smooth nonvanishing functions of their variables.
The class (3) is not normalized but it can be partitioned into two normalized subclasses which are singled out by the conditions, $f_{uu} \neq 0$ and $f_{uu} = 0$, respectively.
The case $f(u) = u^n$, $n \neq 0$ is investigated exhaustively in
[O. Kuriksha, S. Pošta, O. Vaneeva, J. Phys. A: Math. Theor. 47 (2014) 045201].
Other related works:
[M.L. Gandarias, M. Rosa, E. Recio, S.C. Anco, AIP Conference Proceedings 1836 (2017), 020072].
[J. Vašiček, arXiv:1810.02863].

Generalized Kawahara equations

Theorem 1.

The usual equivalence group $G^{\sim}$ of class

$$u_t + \alpha(t) f(u) u_x + \beta(t) u_{xxx} + \sigma(t) u_{xxxxx} = 0$$

consists of the transformations

$$\tilde{t} = T(t), \quad \tilde{x} = \delta_1 x + \delta_2 \int a(t) \mathrm{d}t + \delta_3, \quad \tilde{u} = \delta_4 u + \delta_5,$$

$$\tilde{f} = \delta_0 \left(f + \frac{\delta_2}{\delta_1} \right), \quad \tilde{\alpha}(\tilde{t}) = \frac{\delta_1}{\delta_0 T_t} \alpha(t), \quad \tilde{\beta}(\tilde{t}) = \frac{\delta_1^3}{T_t} \beta(t), \quad \tilde{\sigma}(\tilde{t}) = \frac{\delta_1^5}{T_t} \sigma(t),$$

where δ_j, $j = 0, 1, 2, 3, 4, 5$ are arbitrary constants with $\delta_0 \delta_1 \delta_3 \neq 0$, T is an arbitrary smooth function with $T_t \neq 0$.

Table 1. The group classification of the class $u_t + f(u)u_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0, \quad \beta\sigma \neq 0,\ f \neq u^n$.

	$f(u)$	$\beta(t)$	$\sigma(t)$	Basis of $A^{\max}$
0	$\forall$	$\forall$	$\forall$	∂_x
1	$\forall$	λt^2	δt^4	$\partial_x,\ t\partial_t + x\partial_x$
2	$\forall$	λ	δ	$\partial_x,\ \partial_t$
3	e^u	λt^ρ	$\delta t^{\frac{5\rho+2}{3}}$	$\partial_x,\ 3t\partial_t + (\rho+1)x\partial_x + (\rho-2)\partial_u$
4	e^u	λe^t	$\delta e^{\frac{5}{3}t}$	$\partial_x,\ 3\partial_t + x\partial_x + \partial_u$
5	$\ln u$	$\forall$	$\forall$	$\partial_x, t\partial_x + u\partial_u$
6	$\ln u$	λ	δ	$\partial_x,\ \partial_t,\ t\partial_x + u\partial_u$

$u_t + u^n u_x + \beta(t) u_{xxx} + \sigma(t) u_{xxxxx} = 0$

$\beta(t)$	$\sigma(t)$	Basis of $A^{\max}$
	$n \neq 1$	
$\forall$	$\forall$	∂_x
λt^ρ	$\delta t^{\frac{5\rho+2}{3}}$	∂_x, $3nt\partial_t + (\rho+1)nx\partial_x + (\rho-2)u\partial_u$
λe^t	$\delta e^{\frac{5}{3}t}$	$\partial_x,\ 3n\partial_t + nx\partial_x + u\partial_u$
λ	δ	$\partial_x,\ \partial_t$
	$n = 1$	
$\forall$	$\forall$	$\partial_x,\ t\partial_x + \partial_u$
λt^ρ	$\delta t^{\frac{5\rho+2}{3}}$	$\partial_x,\ t\partial_x + \partial_u$, $3t\partial_t + (\rho+1)x\partial_x + (\rho-2)u\partial_u$
λe^t	$\delta e^{\frac{5}{3}t}$	$\partial_x,\ t\partial_x + \partial_u,\ 3\partial_t + x\partial_x + u\partial_u$
λ	δ	$\partial_x,\ t\partial_x + \partial_u,\ \partial_t$
$\lambda(t^2+1)^{\frac{1}{2}} e^{3\nu \arctan t}$	$\delta(t^2+1)^{\frac{3}{2}} e^{5\nu \arctan t}$	$\partial_x,\ t\partial_x + \partial_u,\ (t^2+1)\partial_t +$ $(t+\nu)x\partial_x + ((\nu - t)u + x)\partial_u$

Conclusions

- Once it is proved that some class is normalized the finding of the equivalence groupoid for its subclasses becomes essentially simpler.
- The study of transformational and normalization properties of classes of PDEs can simplify a lot the further study of their symmetry properties. For classes that are not normalized the method of partition of a class into normalized subclasses works very good.
- Equivalence groupoid can be used in many problems related to classes of DEs: finding exact solutions and conservation laws, study of the integrability and more.

Thank you for your attention!