Sasaki-Ricci flow on the Sasaki-Einstein space $T^{1,1}$

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Outline

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Sasaki manifold and Sasaki potential (1)

A $(2n+1)$-dimensional manifold M is a **contact manifold** if there exists a 1-form η (called a contact 1-form) on M such that

$$\eta \wedge (d\eta)^{n-1} \neq 0 \,.$$

Associated with a contact form η there exists a unique vector field ξ called the **Reeb vector field** defined by the contractions (interior products):

$$\boldsymbol{i}(\xi)\eta = 1 \,,$$

$$\boldsymbol{i}(\xi)d\eta = 0 \,.$$

Sasaki manifold and Sasaki potential (2)

A simple and direct definition of the Sasakian structures is the following:

A compact Riemannian manifold (M, g) is **Sasakian** if and only if its metric cone $(C(M) \cong \mathbb{R}_+ \times M\,,\ \bar{g} = dr^2 + r^2\, g)$ is Kähler.

Here $r \in (0, \infty)$ may be considered as a coordinate on the positive real line $\mathbb{R}_+$. The Sasakian manifold (M, g) is naturally isometrically embedded into the metric cone via the inclusion $M = \{r = 1\} = \{1\} \times M \subset C(M)$.

Let us denote by L_ξ the line subbundle generated by ξ and let $\mathcal{D} = \mathrm{Ker}\eta$ be the contact subbundle in TM. Then we have the following decomposition of the tangent bundle TM of M:

$$TM = \mathcal{D} \oplus L_\xi .$$

Sasaki manifold and Sasaki potential (3)

M can be endowed with a contact structure (Φ, ξ, η), where the endomorphism Φ of the tangent bundle TM is

$$\Phi(X) = \nabla_X \xi \,,$$

for any smooth vector field X on M.

One gets a global 2-form Ω^T on M coming from the contact 1-form η, namely

$$\Omega^T = \frac{1}{2} d\eta.$$

We have that $(\mathcal{D}, \Phi_{|\mathcal{D}}, d\eta)$ gives M a transverse Kähler structure with Kähler form Ω^T defined above and transverse metric g^T given by

$$g^T(X, Y) = d\eta(X, \Phi Y),$$

and related to the Sasaki metric g on M by

$$g = g^T + \eta \otimes \eta.$$

Sasaki manifold and Sasaki potential (4)

We recall that a Riemannian manifold (M, g) that satisfies the Einstein equation

$$Ric_g = \lambda g\,,$$

for a real constant λ (called Einstein constant), where Ric_g stands for the Ricci tensor of the metric g, is said to be an Einstein manifold. Moreover, if the Einstein constant is zero, then the Riemannian space (M, g) is called a Ricci-flat manifold. A Sasaki manifold is said to be a Sasaki-Einstein space if the cone manifold $C(M)$ of M is Kähler Ricci-flat (Calabi-Yau). It is clear that a Sasaki-Einstein space is a Riemannian manifold that is both a Sasaki manifold and an Einstein space.

Notice that the transverse metric associated with a Sasaki-Einstein space is Einstein.

Sasaki manifold and Sasaki potential (5)

Every $(2n+1)$-dimensional Sasakian manifold is locally generated by a free real-valued function K of $2n$ variables, called the Sasaki potential, while every locally Sasaki-Einstein space of dimension $2n+1$ is generated by a locally Kähler-Einstein space of dimension $2n$.

If $\{U_\alpha\}$ is a foliation chart on M with $U_\alpha = I \times V_\alpha$ (where $I \subset \mathbb{R}$ is an open interval and $V_\alpha \subset \mathbb{C}^n$), and $(x, z^1, \ldots, z^n)$ are the local holomorphic coordinates on U_α (with Reeb vector field $\xi = \frac{\partial}{\partial x}$ and $z^1, \ldots, z^n$ are the local holomorphic coordinates on V_α).

Sasaki manifold and Sasaki potential (6)

The **Sasaki potential** *[M. Godliński, W. Kopczyński, P. Nurowski, Class. Quantum Grav.* **17** *(2000) L105-L115]*
K on U_α is chosen in such a way that $\xi(K) = 0$ and

$$\eta = dx + i\sum_{j=1}^{n}(K_{,j}\mathrm{d}z^j) - i\sum_{\bar{j}=1}^{n}(K_{,\bar{j}}d\bar{z}^j),$$

$$d\eta = -2i\sum_{j,\bar{k}=1}^{n} K_{,j\bar{k}}dz^j \wedge d\bar{z}^k,$$

$$g = \eta^2 + 2\sum_{j,\bar{k}=1}^{n} K_{,j\bar{k}}dz^j\mathrm{d}\bar{z}^k$$

$$\phi = -i\sum_{j=1}^{n}[(\partial_j - iK_{,j}\partial_x) \otimes dz^j] + i\sum_{\bar{j}=1}^{n}(\partial_{\bar{j}} + iK_{,\bar{j}}\partial_x) \otimes d\bar{z}^j].$$

Sasaki manifold and Sasaki potential (7)

We recall that a r-form α on M is called **basic** if

$$\iota_\xi \alpha = 0 \quad , \quad \mathcal{L}_\xi \, \alpha = 0 \, ,$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to the vector field ξ. In particular a function φ is basic if and only if $\xi(\varphi) = 0$. In the system of coordinates $(x, z^1, \dots, z^n)$ given above, a basic r-form of type (p, q), $r = p + q$ has the form

$$\alpha = \alpha_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q} \, ,$$

where $\alpha_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q}$ does not depend on x.

Sasaki-Ricci flow (1)

Let M be a smooth manifold equipped with a Sasakian structure (g, η, ξ, Φ). Suppose that we deform the contact form η with a basic function φ as follows:

$$\tilde{\eta} = \eta + d_B^c \varphi,$$

where $d_B^c = \frac{i}{2}(\bar{\partial}_B - \partial_B)$, $d_B = \partial_B + \bar{\partial}_B$ and $\bar{\partial}_B, \partial_B$ denote the basic Dolbeault operators.

The above deformation implies that other fundamental tensors are also modified:

$$\tilde{\Phi} = \Phi - (\xi \otimes (d_B^c \varphi)) \circ \Phi \,,$$
$$\tilde{g} = d\tilde{\eta} \circ (\mathbb{1} \otimes \tilde{\Phi}) + \tilde{\eta} \otimes \tilde{\eta} \,,$$

as well as the transverse form:

$$d\tilde{\eta} = d\eta + d_B d_B^c \varphi.$$

It is known that the quadruplet $(\tilde{g}, \tilde{\eta}, \xi, \tilde{\Phi})$ remains a Sasakian structure on M.

Sasaki-Ricci flow (2)

Let $(g(t), \eta(t), \xi, \Phi(t))$ be a flow having initial data $(g(0), \eta(0), \xi, \Phi(0)) = (g, \eta, \xi, \Phi)$, generated by a basic function $\varphi(t)$ as above and suppose that the basic first Chern class is positive, i.e. $c_B^1 > 0$. Then the **Sasaki-Ricci flow**, also known as **transverse Kähler-Ricci flow** *[K. Smoczyk, G. Wang, Y. Zhang, Intern. J. Math.* **21** *(2010), 951-969; A. Futaki, H. Ono, G. Wang, J. Diff. Geom.* **83** *(2009), 585-635]* is defined by

$$\frac{\partial g^T}{\partial t} = -Ric^T_{g(t)} + (2n+2)g^T(t),$$

where Ric^T is the transverse Ricci curvature.

Sasaki-Ricci flow (3)

Considering a deformation of the Sasaki structure with a basic function φ, in local coordinates the Sasaki-Ricci flow can be expressed as a parabolic Monge-Ampère equation

$$\frac{\partial \varphi}{\partial t} = \log \det(g^T_{j\bar{k}} + \varphi_{j\bar{k}}) - \log(\det g^T_{j\bar{k}}) + (2n+2)\varphi \,.$$

Local coordinates on $T^{1,1}$ (1)

$T^{1,1} = S^2 \times S^3$ is one of the most renowned example of homogeneous Sasaki-Einstein space in dimension five, the standard metric on this manifold being

$$
\begin{aligned}
ds^2(T^{1,1}) =& \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) + \\
& \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 \,,
\end{aligned}
$$

where $\theta_i \in [0, \pi)$, $\phi_i \in [0, 2\pi)$, $i = 1, 2$ and $\psi \in [0, 4\pi)$.

Local coordinates on $T^{1,1}$ (2)

We consider on $T^{1,1}$ a patch of coordinates (ψ, w^1, w^2), where the real coordinates ψ is for the Reeb flow of the Sasaki structure, with

$$\xi = \frac{1}{3}\frac{\partial}{\partial\psi} \, .$$

(z^1, z^2) are transverse complex coordinates addressing the transverse Kähler structure. As on $T^{1,1}$ the transverse structure are locally isomorphic to a product $S^2 \times S^2$, we choose

$$z^1 = \tan\frac{\theta_1}{2} e^{i\phi_1},$$
$$z^2 = \tan\frac{\theta_2}{2} e^{i\phi_2}.$$

Local coordinates on $T^{1,1}$ (3)

We consider the Sasaki potential

$$K = \frac{1}{3}\sum_j \log(1 + z^j \bar{z}^j) - \frac{1}{6}\sum_j \log(z^j \bar{z}^j).$$

For the contact form η we get

$$\begin{aligned}\eta &= \frac{1}{3}d\psi + i\sum_j \frac{\partial K}{\partial z^j}dz^j - i\sum_{\bar{j}} \frac{\partial K}{\partial \bar{z}^j}d\bar{z}^j \\ &= \frac{1}{3}d\psi + \frac{1}{3}\sum_j \cos\theta_j \phi_j.\end{aligned}$$

Transverse Kähler-Ricci flow on $T^{1,1}$ (1)

In the case of the space $T^{1,1}$ the Ricci flow equation has the form:

$$\begin{aligned}\frac{d\varphi}{dt} = & \log\big(\varphi_{1\bar{1}}\varphi_{2\bar{2}} - \varphi_{1\bar{2}}\varphi_{2\bar{1}} \\ & + \cos^4\frac{\theta_1}{2}\varphi_{2\bar{2}} + \cos^4\frac{\theta_2}{2}\varphi_{1\bar{1}} + \cos^4\frac{\theta_1}{2}\cos^4\frac{\theta_2}{2}\Big) \\ & - \log\left(\cos^4\frac{\theta_1}{2}\cos^4\frac{\theta_2}{2}\right) + 6\varphi\,.\end{aligned}$$

Evaluating the derivatives of the basic φ we get:

$$\begin{aligned}\varphi_{j\bar{j}} &= \frac{\partial^2\varphi}{\partial z^j\partial\bar{z}^j} \\ &= \cos^4\frac{\theta_j}{2}\frac{\partial^2\varphi}{\partial\theta_j^2} + \frac{1}{4}\frac{1}{\tan^2\frac{\theta_j}{2}}\frac{\partial^2\varphi}{\partial\phi_j^2} + \frac{1}{2}\cos^2\frac{\theta_j}{2}\frac{1}{\tan\frac{\theta_j}{2}}\cos\theta_j\frac{\partial\varphi}{\partial\theta_j}\,,\end{aligned}$$

with $1 \leq j \leq 2$

Transverse Kähler-Ricci flow on $T^{1,1}$ (2)

$$\varphi_{j\bar{l}} = \frac{\partial^2 \phi}{\partial z^j \partial \bar{z}^l} = \cos^2 \frac{\theta_j}{2} \cos^2 \frac{\theta_l}{2} e^{i(\phi_l - \phi_j)}$$
$$\cdot \left(\frac{\partial^2 \varphi}{\partial \theta_j \partial \theta_l} + \frac{1}{\sin \theta_j \sin \theta_l} \frac{\partial^2 \varphi}{\partial \phi_j \partial \phi_l} - \frac{i}{\sin \theta_j} \frac{\partial^2 \varphi}{\partial \theta_l \partial \phi_j} + \frac{i}{\sin \theta_l} \frac{\partial^2 \varphi}{\partial \phi_l \partial \theta_j} \right)$$

for $i \neq j$.

Transverse Kähler-Ricci flow on $T^{1,1}$ (3)

We search after particular solutions of the transverse Kähler-Ricci flow equation. We factorize the dependences on the variable t and angle coordinates as follows:

$$\varphi(t,\theta_1,\theta_2,\phi_1,\phi_2) = f(t)\cdot g(\theta_1,\theta_2,\phi_1,\phi_2)\,.$$

The Ricci flow equation is still quite involved and searching for some explicit solutions we shall assume that the dependence on the angles (θ_1,ϕ_1), (θ_2,ϕ_2) of the function g separates:

$$g(\theta_1,\theta_2,\phi_1,\phi_2) = g_1(\theta_1,\phi_1) + g_2(\theta_2,\phi_2)\,.$$

With this simplifying assumption the mixed derivatives $\varphi_{1\bar{2}}$ and $\varphi_{2\bar{1}}$ vanish.

Transverse Kähler-Ricci flow on $T^{1,1}$ (4)

Moreover we look for solutions satisfying the following additional constraints:

$$\frac{\partial^2 \varphi}{\partial \theta_1^2} + \frac{1}{\sin^2 \theta_1} \frac{\partial^2 \varphi}{\partial \phi_1^2} + \frac{1}{\tan \theta_1} \frac{\partial \varphi}{\partial \theta_1} = c_1 f(t) \,,$$

$$\frac{\partial^2 \varphi}{\partial \theta_2^2} + \frac{1}{\sin^2 \theta_2} \frac{\partial^2 \varphi}{\partial \phi_2^2} + \frac{1}{\tan \theta_2} \frac{\partial \varphi}{\partial \theta_2} = c_2 f(t) \,,$$

where c_j are some arbitrary real constant.

With these assumptions we get that

$$\varphi_{1\bar{1}} = \cos^4 \frac{\theta_1}{2} \, c_1 \, f(t) \,,$$

$$\varphi_{2\bar{2}} = \cos^4 \frac{\theta_2}{2} \, c_2 \, f(t)$$

Transverse Kähler-Ricci flow on $T^{1,1}$ (5)

Ricci flow equation reduces to an ordinary differential equation for $f(t)$:

$$\frac{df(t)}{dt} \cdot g(\theta_1, \theta_2, \phi_1, \phi_2) = \log\left[f^2(t)(c_1 c_2) + f(t)(c_1 + c_2) + 1\right] + 6f(t) \cdot g(\theta_1, \theta_2, \phi_1, \phi_2)\,.$$

We search for a solution of the form:

$$g(\theta_1, \theta_2, \phi_1, \phi_2) = \frac{1}{2} d_1 \phi_1^2 + h_1(\theta_1) + \frac{1}{2} d_2 \phi_2^2 + h_2(\theta_2)\,,$$

where d_j are some arbitrary real constants.

Transverse Kähler-Ricci flow on $T^{1,1}$ (6)

Functions h_j are

$$h_1(\theta_1) = e_1 \log u_1 - \frac{d_1}{2}(\log u_1)^2 - c_1 \log \sin \theta_1 \,,$$

$$h_2(\theta_2) = e_2 \log u_2 - \frac{d_2}{2}(\log u_2)^2 - c_2 \log \sin \theta_2 \,,$$

where

$$u_j = \frac{\sin \theta_j}{1 + \cos \theta_j} \quad , \quad j = 1, 2 \,.$$

and d_j, e_j are other arbitrary real constants.

Transverse Kähler-Ricci flow on $T^{1,1}$ (7)

The simplest solution is that involving only the constants $e_j \neq 0$ and the rest of the constants is zero. In that case we can state the following proposition:

Proposition

Any metric of the form with arbitrary real constants e_j , $j = 1, 2$

$$\tilde{g} = \frac{1}{9}\left(d\psi + \sum_j (\cos\theta_j + \frac{e_j}{2}) d\phi_j\right)^2 + \frac{1}{6}\sum_j \left(d\theta_j^2 + \sin^2\theta_j d\phi_j^2\right)$$

represents a deformation of the canonical metric on $T^{1,1}$. The deformed contact structure remains Sasaki-Einstein with the contact form

$$\tilde{\eta} = \eta + \frac{1}{6}\sum_j e_j\, d\phi_j = \frac{1}{3}d\psi + \frac{1}{3}\sum_j \cos\theta_j d\phi_j + \frac{1}{6}\sum_j e_j\, d\phi_j\,.$$

Transverse Kähler-Ricci flow on $T^{1,1}$ (8)

A more involved deformation can be obtain assuming that the constants $d_j \neq 0$. In this case we get the following deformation of the Sasaki structure:

Proposition

Any metric of the form with arbitrary real constants d_j , $j = 1, 2$

$$\tilde{g} = \frac{1}{9}\left(d\psi + \sum_j \left(\cos\theta_j + \frac{d_j}{2}\log\tan\frac{\theta_j}{2}\right)d\phi_j + \frac{1}{2}\sum_j d_j \frac{\phi_j}{\sin\theta_j} d\theta_j\right)^2 + \frac{1}{6}\sum_j \left(d\theta_j^2 + \sin^2\theta_j d\phi_j^2\right)$$

represents a deformation of the canonical metric on $T^{1,1}$.

Transverse Kähler-Ricci flow on $T^{1,1}$ (9)

The deformed contact structure remains Sasaki-Einstein with the contact form

$$\tilde{\eta} = \frac{1}{3} d\psi + \frac{1}{3} \sum_j \cos\theta_j d\phi_j + \frac{1}{2} \sum_j d_j \frac{\phi_j}{\sin\theta_j} d\theta_j + \frac{1}{2} \sum_j d_j \log\tan\frac{\theta_j}{2} d\phi_j \, .$$

Transverse Kähler-Ricci flow on $T^{1,1}$ (10)

Let us remark that in both deformations considered above the transverse metric remains unaltered. For $c_1 = c_2 = 0$ the function $f(t)$ satisfying the Ricci flow equation has a very simple solution with the initial condition $f(0) = 0$:

$$f(t) = e^{6t} - 1\,.$$

Transverse Kähler-Ricci flow on $T^{1,1}$ (11)

To summarize we have the following outcome :

Corollary

The families of potential basic functions

$$\varphi_t = (e^{6t} - 1) \sum_j e_j \log z^j \bar{z}^j ,$$

and

$$\varphi_t = (e^{6t} - 1) \Big[\sum_j d_j \log^2 z^i + \frac{1}{2} \sum_j d_j \log z^j \log z^j \bar{z}^j - \frac{1}{4} \sum_j d_j \log^2 z^j \bar{z}^j \Big] ,$$

stand as solutions of the transverse Kähler-Ricci flow equation on the manifold $T^{1,1}$.

Transverse Kähler-Ricci flow on $T^{1,1}$ (12)

Finally, let us consider deformations of the Sasaki structures involving the constants $c_j \neq 0$. In this case we have a modification of the transverse metric as follows:

Proposition

The deformed contact structure with the contact form

$$\tilde{\eta} = \eta + \sum_j c_j \cos\theta_j d\phi_j = \frac{1}{3}\big(d\psi + (1-3c_j)\cos\theta_j d\phi_j\big)\,.$$

remains Sasaki with the metric

$$\begin{aligned}\tilde{g} =& \frac{1}{9}\Big[d\psi + \sum_j (1-3c_j)\cos\theta_j d\phi_j\Big]^2 \\ &+ \frac{1}{6}\sum_j (1+3c_j)\big(d\theta_j^2 + \sin^2\theta_j d\phi_j^2\big)\,.\end{aligned}$$

Outlook

- Killing forms on deformed manifolds under Sasaki-Ricci flow
- Integrals of motion on deformed Sasaki-Einstein spaces
- Sasaki-Ricci flow on $Y^{p,q}$
- Sasaki-Ricci flow on 3-Sasakian manifolds

Appendix (1)

$$\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z^j} dz^j \, ,$$

$$\bar{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}^j} d\bar{z}^j \, .$$

$$Ric^T(X, Y) = Ric(X, Y) + 2g^T(X, Y) \, .$$

Let $\rho^T = Ric^T(\Phi\cdot, \cdot)$ and $\rho = Ric(\Phi\cdot, \cdot)$.

ρ^T is called the *transverse Ricci form*.

$$\rho^T = \rho + 2d\eta \, .$$

Appendix (2)

Transverse Einstein metric

$$Ric^T = cg^T \, .$$

ρ^T is a closed basic form and its basic cohomology class $[\rho^T]_B = c_B^1$ is the basic first Chern class.

c_B^1 is called *positive* (respectively, *negative, null*) if it contain a *positive* (respectively, *negative, null*) representation

$$c_B^1 = k[d\eta]_B$$

where $k = +1, -1, 0$.