The Null-Time-like Boundary Problems of Linear Wave Equation in Asymptotically Anti-de Sitter Space-time

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Jointed work with Dr. L. Zhang

- Cauchy problem for linear wave equation

Theorem : If L is a second order hyperbolic operator with C^{r+1} coefficients,

$$Lu = 0$$

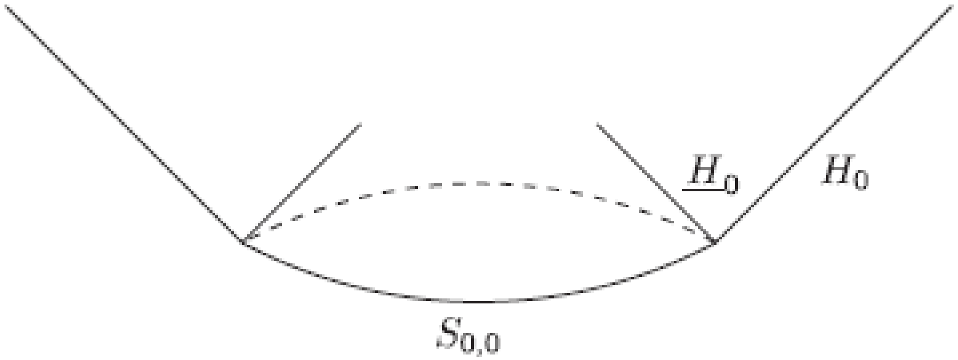
$$u(0,x) = \phi(x) \in H^r(\Sigma_0), \quad u_t(0,x) = \psi(x) \in H^{r-1}(\Sigma_0), (r \geq 1)$$

Then exists unique solution $u(t,x)$ and

1. $\partial_t^j u \in H^{r-j}(\Sigma_t)$ for any t, $j \leq r$;
2. $\sum_{j=0}^{r} ||\partial_t^i u||_{r-j}^2 \leq C(||\phi||_r^2 + ||\psi||_{r-1}^2)$

- Similar result holds for Einstein equations (Y. Choquet-Bruhat, 1950’)

- Characteristic initial value problem

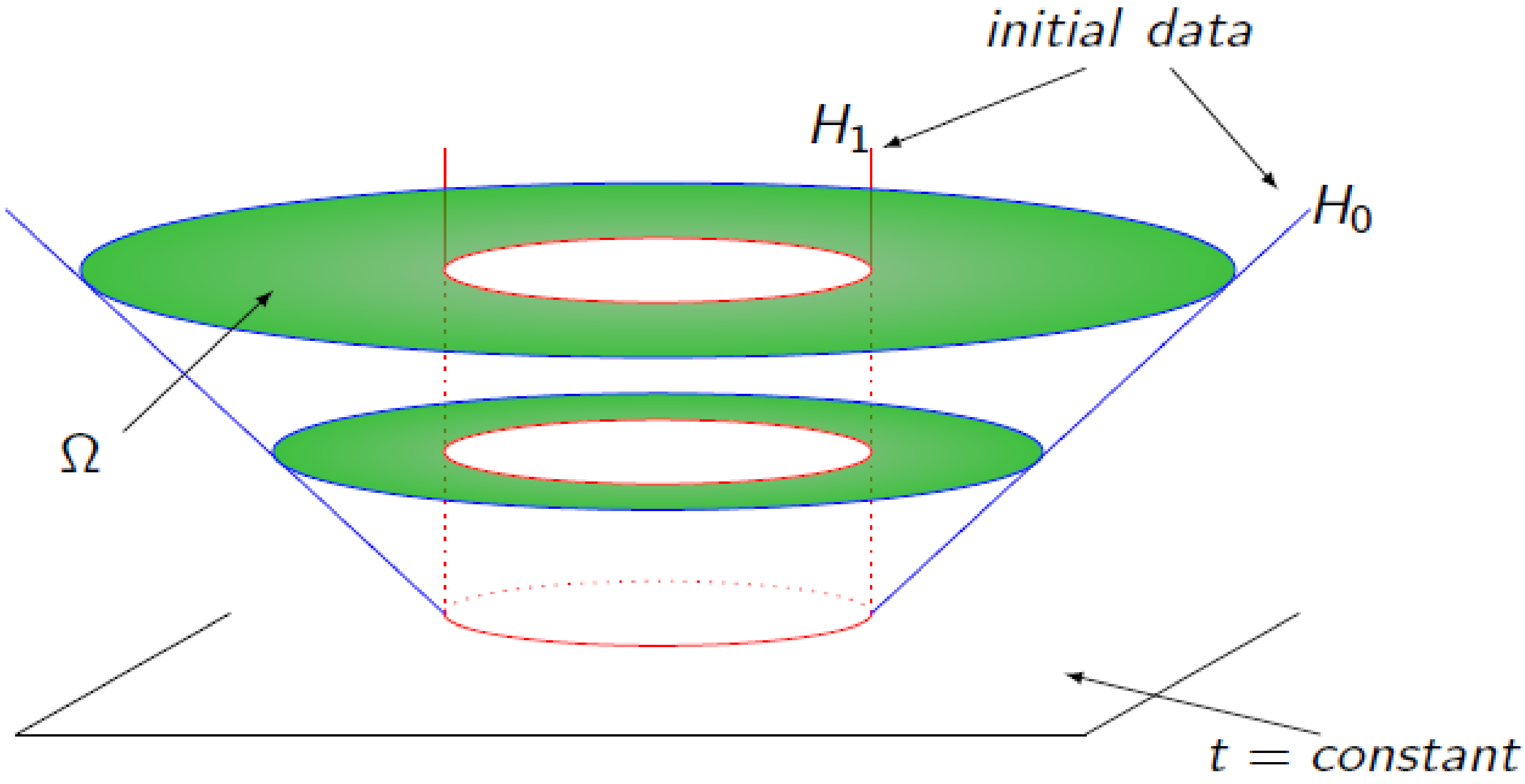


Impose initial data on two intersected null hypersurfaces.

1. Local existence : Z. Hagen, H. J. Seifert, 1977; H. Friedrich, 1980; A. Rendall, 1990
2. Finite region : J. Luk, 2011
3. Semi-global existence : X.-p. Zhu and J.-b. Li, 2016

- Null-time-like boundary value problem

Impose initial and boundary data on intersected null hypersurface and time-like hypersurface.



1. R. Bartnik, 1996 : quasi spherical gauge.
2. R.Balean, 1997: linear wave equation in Minkowski space
3. R.Balean, R.Bartnik, 1998 : Maxwell equations in Minkowski space
4. Q. Han, L. Zhang, 2017 : linear wave equation in asymptotic flat space-time

- Physical Motivation : AdS/CFT correspondence

1. Investigate AdS/CFT in terms of boundary value problem (Witten, 1998, 2018)

2. Holographic model of condensed matter, “3H” model (Hartnoll, Herzog and Horowitz, PRL, 2008)

3. Entropy production for holography (Bredberg, Keeler, Lysov and Strominger, JHEP, 2011; Tian, Wu and Zhang, JHEP, 2012)

4. Dynamical process with dissipation in holographic method

- Asymptotic AdS space-time

$$g = -Ve^{2\eta}\mathrm{d}\tau^2 - 2e^{2\eta}\mathrm{d}\tau\mathrm{d}r + r^2 h_{AB}(\mathrm{d}x^A - U^A\mathrm{d}\tau)(\mathrm{d}x^B - U^B\mathrm{d}\tau),$$

$$V = 1 - \frac{\Lambda}{3}e^{\zeta}r^2 + O(\frac{1}{r}), \quad \Lambda < 0,$$

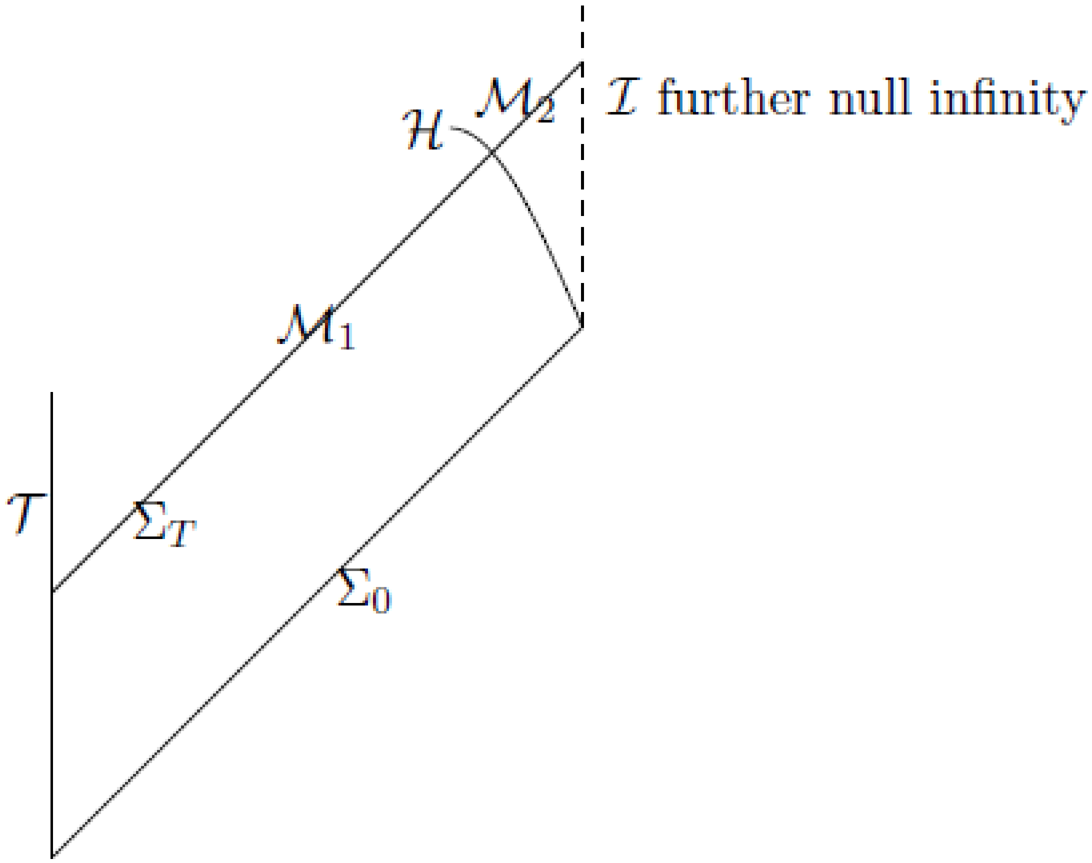
$$\lim_{r\to\infty} rU^A = \lim_{r\to\infty} \zeta = \lim_{r\to\infty} \eta = 0,$$

$$h_{AB}\mathrm{d}x^A\mathrm{d}x^B \to g_{S^2},$$

Introduce new coordinates $(\tau, z = \frac{1}{r}, x_2, x_3),$

Conformal boundary : $r \to \infty$ (z = 0)

Region considered : $\Omega_T = \{(\tau, z) | 0 < \tau < T, 0 < z < \frac{1}{R}\} \times S^2,$



- Equation :

$$\Box_g u = 0,$$

- Boundary condition :

$$u|_{\Sigma_0=\{\tau=0\}} = \frac{\varphi e^{-\eta}}{r},$$

$$u|_{\mathcal{T}=\{r=R\}} = \frac{\psi_1 e^{-\eta}}{R},$$

$$\lim_{r\to\infty} e^{\eta} r u = \psi_2, \qquad \text{for the given functions } \psi_2 \text{ on } [0,+\infty)\times S^2.$$

- Boundary of the region :

$$\Sigma_0 = \{(0,z)|0 \le z \le \frac{1}{R}\} \times S^2, \qquad \mathcal{T} = \{(\tau, \frac{1}{R})|0 \le \tau \le T\} \times S^2,$$

$$\Sigma_T = \{(T,z)|0 \le z \le \frac{1}{R}\} \times S^2, \qquad \mathcal{I} = \{(\tau,0)|0 \le \tau < +\infty\} \times S^2,$$

- Conformal method

$$\hat{g} = \Xi^2 g = -z^2 V d\tau^2 + 2d\tau dz + h_{AB} e^{-2\eta}(dx^A - U^A d\tau)(dx^B - U^B d\tau),$$

where $\Xi = ze^{-\eta}$.

then

$$\Box_g u = 0 \Rightarrow \Box_{\hat{g}} v + \omega v = 0,$$

$$\omega = \frac{1}{2z}(\partial_A \gamma)U^A + \frac{1}{z}\partial_A U^A - \Box_{\hat{g}}\eta - 3\hat{g}^{ij}\partial_i\eta\partial_j\eta,$$

$$\gamma = det(h_{AB}), \qquad u = \Xi v$$

Boundary condition :

$$v|_{\Sigma_0} = \varphi,$$
$$v|_{\mathcal{T}} = \psi_1,$$
$$v|_{\mathcal{I}} = \psi_2.$$

- Null tertad

$$N_1 = \nabla\tau = \partial_z, \quad N_2 = -(\nabla z - \frac{1}{2}z^2 V\nabla\tau) = -(\partial_\tau + \frac{1}{2}z^2 V\partial_z + g^{1A}\partial_A),$$

$$\hat{g}(N_1, N_1) = \hat{g}(N_2, N_2) = 0, \quad \hat{g}(N_1, N_2) = -1.$$

for any fixed (τ, z), denote correspondent S^2 as $S^2_{(\tau,z)}$; denote H_μ for the in-going null hypersurface generated by N_2 and starts at $S^2_{(0,\mu)}$.

$$H_{(\nu,\frac{1}{R})} = \bigcup_{\mu\in(\nu,\frac{1}{R})} H_\mu, \qquad \Omega_{T,\nu} = \Omega_T \cap H_{(\nu,\frac{1}{R})}.$$

$$\mathcal{H} := H_0 \qquad \mathcal{M}_1 = H_{(0,\frac{1}{R})}.$$

$\mathcal{M}_2$ *for the domain bounded by* $\mathcal{H}$ *and* $\mathcal{I}$.

- Main theorem : (Wu and Zhang, 2019)

Theorem 1.1. *Suppose g sufficiently regular and with conformal regular condition (see later). Assume* $\psi_1 \in H^{2k}([0,T] \times S^2)$, $\psi_2 \in H^k([0,T] \times S^2)$, *and*

$$\|\varphi\|_{\widetilde{H}^{2k}((R,\infty)\times S^2)} = \sum_{|\alpha|\leq 2k, \alpha=(\alpha_1,\alpha_2,\alpha_3)} \left(\int_R^\infty \int_{S^2} \frac{|\partial^\alpha \varphi|^2}{r^{2+4\alpha_1}} dr d\Sigma\right)^{\frac{1}{2}} < +\infty.$$

Then, there exists a unique solution u, which satisfies (1.5)-(1.6)-(1.7) *in* $[0,T]\times[R,+\infty)\times S^2$, *and*

$$\|e^\eta r u\|_{\widetilde{H}^k([0,T]\times(R,\infty)\times S^2)} = \sum_{|\alpha|\leq k, \alpha=(\alpha_0,\alpha_1,\alpha_2,\alpha_3)} \left(\int_0^T \int_R^\infty \int_{S^2} \frac{|\partial^\alpha(e^\eta r u)|^2}{r^{2+4\alpha_1}} d\Sigma dr d\tau\right)^{\frac{1}{2}} \tag{1.8}$$
$$\leq C\{\|\varphi\|_{\widetilde{H}^{2k}((R,\infty)\times S^2)} + \|\psi_1\|_{H^{2k}([0,T]\times S^2)} + \|\psi_2\|_{H^k([0,T]\times S^2)}\},$$

where C is a constant depending on g and k.

Sketch of Proof :

- Local existence of solution

  Analytic approximation method

1. Consider problem for analytic coefficients and initial data.

2. Using analytic function to approximate general functions with the help of energy estimate.

✓ Analytic case :

Consider equations in the form $\Box_{\hat{g}} v + \omega v = 0,$

with all coefficients are analytic. The initial data on time-like and null boundary is also analytic.

$$2\partial_{z\tau} v + z^2 V \partial_{zz} v + 2g^{1A}\partial_{zA} v + g^{AB}\partial_{AB} v + a^i \partial_i v + bv = 0.$$

$$\psi = \sum_{i=0}^{\infty} \psi_i(\theta)\tau^i.$$

define

$$u^0 = v = \sum_{i=0}^{\infty} u_i^0(z,\theta)\tau^i,$$

$$u^1 = \partial_z v = \sum_{i=0}^{\infty} u_i^1(z,\theta)\tau^i,$$

$$u^A = \partial_A v = \sum_{i=0}^{\infty} u_i^A(z,\theta)\tau^i, \ \ A = 2,3,$$

$$w = \partial_\tau v = \sum_{i=0}^{\infty} w_i(z,\theta)\tau^i.$$

Equation becomes

$$
\begin{aligned}
\partial_\tau u^0 &= w, \\
2\partial_\tau u^1 &= -z^2 V \partial_z u^1 - 2g^{1A}\partial_z u^A - g^{AB}\partial_B u^A + \bar{a}^i u^i + cw, \\
\partial_\tau u^A &= \partial_A w, \\
2\partial_z w &= -z^2 V \partial_z u^1 - 2g^{1A}\partial_z u^A - g^{AB}\partial_B u^A + \bar{a}^i u^i + cw.
\end{aligned}
$$

Algebraic calculation implies

$$
\begin{aligned}
(i+1)u^0_{i+1} &= w_i, \\
2(i+1)u^1_{i+1} &= \sum_{k\le i} L_k[\partial u^0_k, \partial u^1_k, \partial u^2_k, \partial u^3_k, w_k], \\
(i+1)u^A_{i+1} &= \partial_A w_i, \\
2\partial_z w_i + dw_i &= \sum_{k\le i} L_k[\partial u^0_k, \partial u^1_k, \partial u^2_k, \partial u^3_k] + \sum_{k\le i-1} L_k[w_k].
\end{aligned}
$$

Solving the ODE for w_i , one can get all Taylor coefficients of u^i and w, which means one get formal solution of the analytic system.

Using majorizing function method, one can prove above formal solution converge.

Since the region is closed, one can use analytic function to approximate the smooth function. Now one needs to show if initial data $P^i \to \varphi$ and $Q^i \to \psi$ in H^{2k} space, and equation coefficients also satisfies $g^i \to g$ and $\omega^i \to \omega$, whether the associated solution u^iwill converge to a solution u ?

✓ Enenrgy estimate

“Energy-momentum tensor” for C^1 function,

$$Q[\phi](X,Y) = (X\phi)(Y\phi) - \frac{1}{2}g(X,Y)|\nabla\phi|^2 - g(X,Y)\phi^2.$$

One can construct current associated with vector X as

$$\mathrm{div}(hP[\phi,X]) = (\Box_g\phi)(hX\phi) + \frac{1}{2}hQ[\phi]_{\alpha\beta}\ {}^{(X)}\pi^{\alpha\beta} + Q[\phi](\nabla h, X) - 2g(X,\nabla\phi)h\phi,$$

$$P[\phi,X]_\alpha = Q[\phi]_{\alpha\beta}X^\beta, \quad {}^{(X)}\pi^{\alpha\beta} = \partial^\alpha X^\beta + \partial^\beta X^\alpha - X(g^{\alpha\beta}).$$

Carefully choose vector field X and weight function h, integral above eqution on $\Omega_{T,0}$ and integral by part for left side, one can get if

$$\mathcal{L}v = \Box_g v + a^i\partial_i v + \omega v = f.$$

Theorem 3.9. *For some fixed* $T > 0$, *and* v *satisfies* (3.8). *Then, there exists constants* $q_0 > 0, l > 0$ *depending on* $|g^{ij}|_{C^1(\Omega_T)}$, $|a^i|_{L^\infty(\Omega_T)}$ *and* $|\omega|_{L^\infty(\Omega_T)}$, *such that, for any* $q > q_0$, *and* $h = e^{-lq\tau + qz}$, [1]

$$q^{\frac{1}{2}}||v||_{H^1_h(\Omega_{T,0})} + ||\partial_z v||_{L^2_h(\mathcal{T})} + ||v||_{H^1_h(\Sigma_T)} + \int_{\mathcal{H}} hE_{\mathcal{H}}[v,1,1]d\mathcal{H} \tag{3.10}$$
$$\leq C\{||v||_{H^1_h(\Sigma_0)} + ||v||_{H^1_h(\mathcal{T})} + ||f||_{L^2_h(\Omega_{T,0})}\},$$

where C *is a constant depending on* $|g^{ij}|_{C^1(\Omega_T)}$, $|a^i|_{L^\infty(\Omega_T)}$ *and* $|\omega|_{L^\infty(\Omega_T)}$.

Where

$$||u||_{H^p_h(\Omega)} = \sum_{i=0}^{p} ||h^{\frac{1}{2}}\nabla^i u||_{L^2(\Omega)},$$

$E_{\mathcal{H}}[\phi, b_1, b_2]$ is surface term on cosmologic horizon *H*

Higher order derivative estimate :

$$[\Box, X]\phi = \pi^{\alpha\beta}\nabla^2\phi_{\alpha\beta} + \nabla_\alpha\pi^{\alpha\beta}\partial_\beta\phi - \frac{1}{2}\partial^\alpha(tr\pi)\partial_\alpha\phi.$$

Which implies for any $|\alpha| = p$,

$$\Box_g(\partial^\alpha v) = \sum_{|\beta|=p+1} c_{\alpha\beta}\partial^\beta v + f_\alpha, \qquad f_\alpha = \sum_{|\beta|\le p} c_{\alpha\beta}\partial^\beta v.$$

Then

Theorem 3.14. *For any $p \ge 1$, there exists q_0 depending on $|g^{ij}|_{C^1(\Omega_T)}$, $|a^i|_{L^\infty(\Omega_T)}$, $|\omega|_{L^\infty(\Omega_T)}$ and p, and l the same as in theorem 3.9, then, for $q \ge q_0$ and $h = e^{-ql\tau + qz}$, we have*

$$\sum_{|\alpha|=p+1} (q^{\frac{1}{2}}||\partial^\alpha v||_{L^2_h(\Omega_{T,0})} + ||\partial^\alpha v||_{L^2_h(\mathcal{T})} + \int_{\mathcal{H}} hE_{\mathcal{H}}[\partial^\alpha v, 1, 1]d\mathcal{H}) \tag{3.22}$$
$$\le C\{||\varphi||_{H_h^{(2p+2)}(\Sigma_0)} + ||\psi_1||_{H_h^{2p+2}(\mathcal{T})} + ||f||_{H_h^{2p+1}(\Omega_{T,0})}\},$$

where C is constant depending on $|g^{ij}|_{C^2(\Omega_T)}$, $|a^i|_{C^1(\Omega_T)}$ and $|\omega|_{L^\infty(\Omega_T)}$.

Finally, one has

Corollary 3.17. *we have*

$$||v||_{H^k(\mathcal{M}_1\cap\Omega_T)} \leq C\{||\varphi||_{H^{2k}(\Sigma_0)} + ||\psi_1||_{H^{2k}(\mathcal{T})}\}, \tag{3.31}$$

where C *is a constant depending on* $|g^{ij}|_{C^{2k-1}(\Omega_T)}$, $|\omega|_{C^{2k-2}(\Omega_T)}$, k, T, *and* R.

✓ Estimate on M_2

Since in Theorem 3.9 and 3.14, one has got the control for the data on H , one can use similar idea to control the data in M_2 in terms of the data on H and conformal boundary.

Theorem 3.18. *There exists q_0 and l depending on $|g^{ij}|_{C^1(\mathcal{M}_2\cap\Omega_T)}$, $|\omega|_{L^\infty(\mathcal{M}_2\cap\Omega_T)}$, such that, for any $q \geq q_0$, and $h = e^{-ql\tau+qz}$,*

$$\begin{aligned} &q^{\frac{1}{2}}||v||_{H^1_h(\Omega_T\cap\mathcal{M}_2)} + ||v||_{H^1_h(\Sigma_T\cap\mathcal{M}_2)} + ||\partial_z v||_{L^2_h(\mathcal{I})} \\ \leq &C\{\int_{\mathcal{H}} hE_{\mathcal{H}}[v,1,1]d\mathcal{H} + ||v||_{H^1_h(\mathcal{I})} + ||f||_{L^2_h(\mathcal{M}_2\cap\Omega_T)}\}, \end{aligned} \tag{3.32}$$

where C is a constant depending on $|g^{ij}|_{C^1(\mathcal{M}_2\cap\Omega_T)}$, $|\omega|_{L^\infty(\mathcal{M}_2\cap\Omega_T)}$ and Λ.

Theorem 3.19. *For any α with $|\alpha| = k \geq 2$, there exists q_0 depending on k, $|g^{ij}|_{C^1(\mathcal{M}_2\cap\Omega_T)}$, $|\omega|_{L^\infty(\mathcal{M}_2\cap\Omega_T)}$, and l depending on $|g^{ij}|_{C^1(\mathcal{M}_2\cap\Omega_T)}$, $|\omega|_{L^\infty(\mathcal{M}_2\cap\Omega_T)}$, such that, for any $q \geq q_0$, and $h = e^{-ql\tau+qz}$,*

$$\begin{aligned}&\sum_{|\alpha|=k}[q^{\frac{1}{2}}||\partial^\alpha v||_{H^1_h(\Omega_T\cap\mathcal{M}_2)} + ||\partial^\alpha v||_{L^2_h(\mathcal{I})}]\\ \leq & C\{\sum_{|\alpha|\leq k}\int_{\mathcal{H}} hE_{\mathcal{H}}[\partial^\alpha v,1,1]d\mathcal{H} + ||\psi_2||_{H^k_h(\mathcal{I})} + ||f||_{H^{k-1}_h(\mathcal{M}_2\cap\Omega_T)}\},\end{aligned} \tag{3.35}$$

where C is a constant depending on k, $|g^{ij}|_{C^k(\mathcal{M}_2\cap\Omega_T)}$, $|\omega|_{C^{k-1}(\mathcal{M}_2\cap\Omega_T)}$ and Λ.

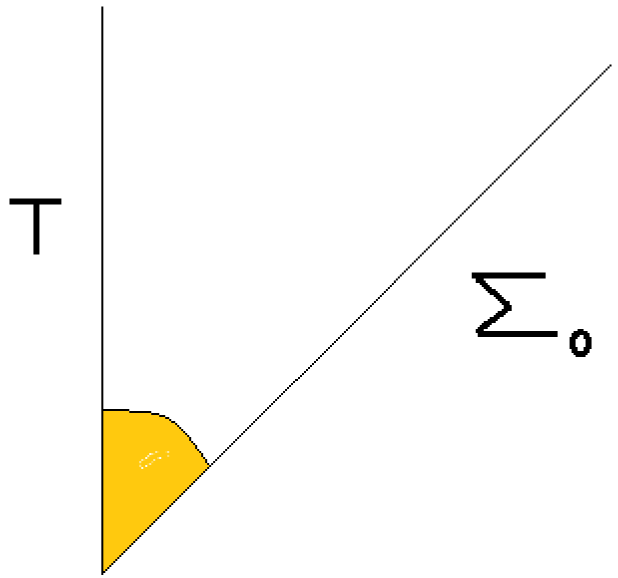
Corollary 3.20. *Suppose v is a solution of (3.1)-(3.2), then, for any $k \geq 1$, we have*

$$||v||_{H^k(\Omega_T)} \leq C\{||\varphi||_{H^{2k}(\Sigma_0)} + ||\psi_1||_{H^{2k}(\mathcal{T})} + ||\psi_2||_{H^k(\mathcal{I})}\}, \tag{3.36}$$

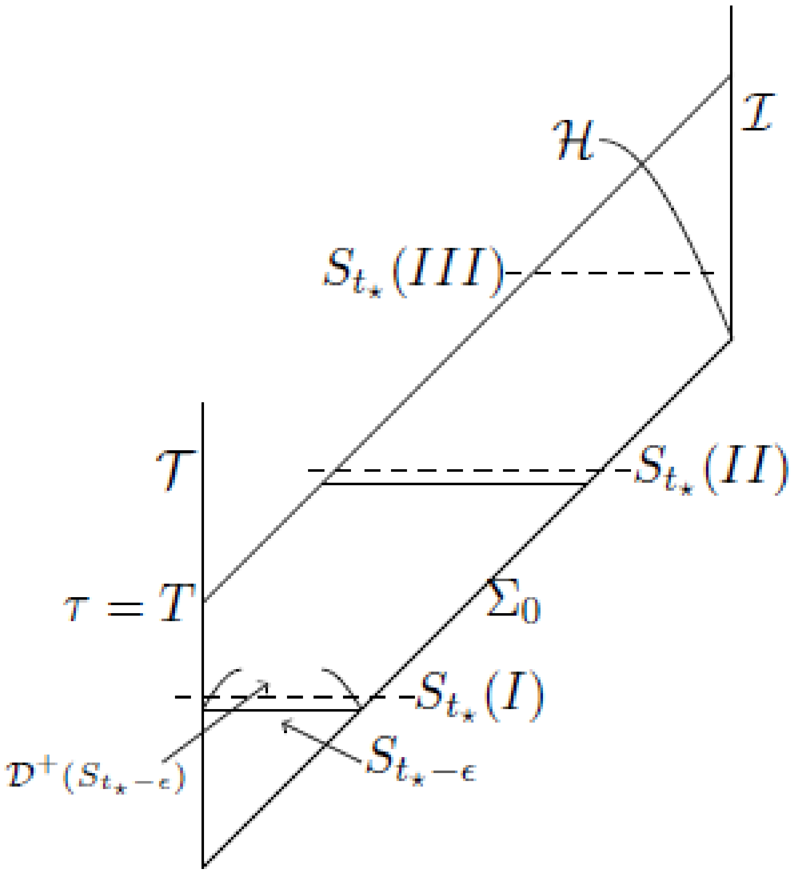
where C is a constant depending on $|g^{ij}|_{C^{2k-1}(\Omega_T)}$, $|\omega|_{C^{2k-2}(\Omega_T)}$, k, Λ, T, and R.

Since P^i , Q^i , g^i , ω^i are all Cauchy sequences, with the control of corollary 3.17 and 3.20, u^i is a Cauchy sequence in $H^k(O(S_{(0,1/R)}))$ obviously, so $u^i \to u$ and u is a solution of equation (Here we use the linearity of equation).

So one get the local existence of wave equation in some neighborhood of $S_{(0,1/R)}$.



- Global existence (Bootstrap method)



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Then, there exists a unique solution u, which satisfies (1.5)-(1.6)-(1.7) in $[0,T]\times[R,+\infty)\times S^2$, and

$$\begin{aligned}\|e^\eta r u\|_{\widetilde{H}^k([0,T]\times(R,\infty)\times S^2)} &= \sum_{|\alpha|\leq k,\alpha=(\alpha_0,\alpha_1,\alpha_2,\alpha_3)} \Big(\int_0^T \int_R^\infty \int_{S^2} \frac{|\partial^\alpha (e^\eta r u)|^2}{r^{2+4\alpha_1}} d\Sigma dr d\tau\Big)^{\frac{1}{2}} \\ &\leq C\{\|\varphi\|_{\widetilde{H}^{2k}((R,\infty)\times S^2)} + \|\psi_1\|_{H^{2k}([0,T]\times S^2)} + \|\psi_2\|_{H^k([0,T]\times S^2)}\},\end{aligned} \tag{1.8}$$

where C is a constant depending on g and k.

Discussion :

1. null-time-like boundary value problem of mass-less scalar field is proved, free data is the value of field on boundary and conformal boundary.
2. This result can be generalized to massive field, at least the field mass is small enough.
3. Similar result holds for Maxwell field.
4. Linear gravity is uneasy, since one need to deal with the gauge freedom.
5. Since asymptotic AdS space-time allows negative mass, it is interesting whether the negative mass will destroy these estimates ?
6. Final aim : how about for Einstein equation ?

THANK YOU