Construction of vacuum initial data by the conformally covariant split system

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- There is another way to construct vacuum initial data, referred to as 'the conformally covariant split' or, historically, 'Method B.'
- Joint with P. Mach and Y. Wang, we prove existence of solutions of the conformally covariant split system giving rise to non-constant mean curvature vacuum initial data for the Einstein field equations.

Spacetime and the Einstein Field Equations

- Let $(N^{1,3}, \hat{g})$ be a Lorentz manifold satisfying the vacuum Einstein field equations

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- Let $(M^3, \widetilde{g}_{ij}, K_{ij})$ be a spacelike hypersurface in $(N^{1,3}, \hat{g})$. Here $\widetilde{g}_{ij}$ is the induced 3-metric of M and K_{ij} is the second fundamental form of M in N.

Vacuum Constraint Equations

- The triple $(M, \widetilde{g}, K)$ satisfies the *vacuum Einstein's constraints*.

$$\widetilde{R} - |K|^2_{\widetilde{g}} + (\mathrm{tr}_{\widetilde{g}} K)^2 = 0 \quad \text{(Hamiltonian cosntraint)}, \tag{2a}$$

$$\mathrm{div}_{\widetilde{g}} K - d\mathrm{tr}_{\widetilde{g}} K = 0 \quad \text{(momentum constraint)}, \tag{2b}$$

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  where $\widetilde{R}$ is the scalar curvature of M with respect to the metric $\widetilde{g}$.
- These equations are coming from (the contracted version of) the Gauss-Codazzi-Mainardi equations in submanifold geometry. (Necessary conditions)

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- This problem is *notoriously difficult*!
- There is a so-called conformal method. (Lichnerowicz, Choquet-Bruhat, York, Isenberg, ...)

Conformal Method - A

- Free data (M^3, g, σ, τ):
  g - a Riemannian metric on M; σ - a symmetric trace- and divergence-free (TT) tensor of type $(0, 2)$; τ a smooth function on M.

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- Consider the following system of equations for a positive function ϕ and a one-form W:

$$-8\Delta\phi + R\phi = -\frac{2}{3}\tau^2\phi^5 + |\sigma + LW|^2\phi^{-7}, \tag{3a}$$

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- Here $\Delta = \nabla_i\nabla^i$ and R are the Laplacian and the scalar curvature computed with respect to metric g, and $\Delta_L W$ is defined as $\Delta_L W = \mathrm{div}_g(LW)$, where L is the conformal Killing operator,

$$(LW)_{ij} = \nabla_i W_j + \nabla_j W_i - \frac{2}{3}(\mathrm{div}_g W)g_{ij}. \tag{4}$$

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- System (3) is referred to as the *vacuum conformal constraints*.
- A dual to a form W satisfying the equation $LW = 0$ is called a conformal Killing vector field.
- **Proposition A.** Suppose that a pair (ϕ, W) solves the vacuum conformal constraints (3). Define $\widetilde{g} = \phi^4 g$, and $K = \frac{\tau}{3}\phi^4 g + \phi^{-2}(\sigma + LW)$. Then the triple $(M, \widetilde{g}, K)$ becomes an initial data set satisfying the *vacuum Einstein's constraints* and $\mathrm{tr}_{\widetilde{g}} K = \tau$.

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- The following table [Isenberg] summarizes whether or not the Lichnerowicz equation admits a positive solution.

	$\sigma^2 \equiv 0, \tau = 0$	$\sigma^2 \equiv 0, \tau \neq 0$	$\sigma^2 \not\equiv 0, \tau = 0$	$\sigma^2 \not\equiv 0, \tau \neq 0$
$\mathcal{Y}^+$	No	No	Yes	Yes
$\mathcal{Y}^0$	Yes	No	No	Yes
$\mathcal{Y}^-$	No	Yes	No	Yes

Here $\mathcal{Y}$ denotes the Yamabe constant. For data in the class $(\mathcal{Y}^0, \sigma^2 \equiv 0, \tau = 0)$, any constant is a solution. For data in all other classes for solutions exist, the solution is unique.

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- Some results are obtained when $d\tau/\tau$ or σ are small.
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- Recently, Dahl, Gicquaud, and Humbert proved the following criterion for the existence of solutions to Eqs. (3). Assume that (M, g) has no conformal Killing vector fields and that $\sigma \not\equiv 0$, if the Yamabe constant $Y(g) \geq 0$. Then, if the *limit equation*

$$\Delta_L W = \alpha\sqrt{\frac{2}{3}}|LW|\frac{d\tau}{\tau} \tag{5}$$

  has no nonzero solutions for all $\alpha \in (0, 1]$, the vacuum conformal constraints (3) admit a solution (ϕ, W) with $\phi > 0$.

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- Moreover, they provided an example on the sphere $\mathbb{S}^3$ such that the limit equation (5) does have a nontrivial solution for some $\alpha_0 \in (0, 1]$.

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- Unfortunately, the result of Dahl, Gicquaud, and Humbert is not an alternative criterion. In fact, Nguyen found that there also exists an example such that both the limit equation (5) and the vacuum conformal constraints (3) have nontrivial solutions.

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- The Lichnerowicz equation (3a) has a covariance property under conformal changes of the metric g. Namely, if ϕ is a solution of (3a) and ψ is any positive function, one may define $\widetilde{g} = \psi^4 g$, $\widetilde{\omega} = \psi^{-6}\omega$, $\widetilde{\phi} = \psi^{-1}\phi$, then

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- But the vector equation (3b) does not possess such a property.

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- We are trying to find a positive function ϕ and a one-form W satisfying the so-called 'conformally covariant split system:'

$$\Delta\phi - \frac{1}{8}R\phi + \frac{1}{8}|\sigma|^2\phi^{-7} + \frac{1}{4}\langle\sigma, LW\rangle\phi^{-1} - \left(\frac{1}{12}\tau^2 - \frac{1}{8}|LW|^2\right)\phi^5 = 0, \tag{7a}$$

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- **Proposition B.** Let $\tilde{g} = \phi^4 g$, and $K = \frac{\tau}{3}\phi^4 g + \phi^{-2}\sigma + \phi^4 LW$. For (ϕ, W) solving system (7), the triple $(M, \widetilde{g}, K)$ becomes vacuum initial data.

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- Then the system (7) can be written as

$$\phi^{-5} P_{g,\omega}\phi = \frac{2}{3}\tau^2, \tag{8a}$$

$$\Delta_{g,\phi} W = \frac{2}{3} d\tau, \tag{8b}$$

where $\Delta_{g,\phi} W = \phi^{-6} \mathrm{div}_g(\phi^6 L_g W)$.

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- We now make the following conformal change: $\widetilde{g} = \psi^4 g,\ \widetilde{\phi} = \psi^{-1}\phi,\ \widetilde{\sigma} = \psi^{-2}\sigma,\ \widetilde{W} = \psi^4 W.$

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- Further, we have $\widetilde{\sigma}$ is still TT with respect to the metric $\widetilde{g}$ and $\widetilde{\omega} = \widetilde{\omega}(\widetilde{\sigma}, \widetilde{\phi}, \widetilde{W}, \widetilde{g}) = \psi^{-6}\omega$.

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- Then for the corresponding vector equation, we now have $\Delta_{\widetilde{g},\widetilde{\phi}}\widetilde{W} = \Delta_{g,\phi}W$.
- The operator given by

$$\mathcal{P}_g \begin{pmatrix} \phi \\ W \end{pmatrix} := \begin{pmatrix} \phi^{-5} P_{g,\omega}\phi \\ \Delta_{g,\phi} W \end{pmatrix} \tag{9}$$

is conformally covariant, i.e.,

$$\mathcal{P}_{\widetilde{g}} \begin{pmatrix} \widetilde{\phi} \\ \widetilde{W} \end{pmatrix} = \mathcal{P}_g \begin{pmatrix} \phi \\ W \end{pmatrix} = \frac{2}{3} \begin{pmatrix} \tau^2 \\ d\tau \end{pmatrix}. \tag{10}$$

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- We use the implicit function theorem to deduce existence of other solutions of Eqs. (7) with $\tau \neq \mathrm{tr}_g K$.
- An immediate observation concerning system (7) is that it admits the following scaling symmetry. Suppose that system (7) has a solution (ϕ, W). Set $\hat{\phi} = \mu^{-\frac{1}{4}} \phi$, $\hat{W} = \mu^{\frac{1}{2}} W$ for some positive number $\mu \in \mathbb{R}^+$. Then $(\hat{\phi}, \hat{W})$ satisfy system (7) with the data $\hat{\sigma}$ and $\hat{\tau}$ given by $\hat{\sigma}_{ij} = \mu^{-1} \sigma_{ij}$, $\hat{\tau} = \mu^{\frac{1}{2}} \tau$.

Conformally Covariant Split System on a Closed Manifold

- Assume that (M, g) is a closed 3-dimensional Riemannian manifold. Making use of the implicit function theorem, we construct a family of solutions of the conformally covariant split system (7) on M. These solutions give rise to vacuum initial data.

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- **Theorem 1.** Suppose that we already have vacuum initial data (M, g, K). Assume that $\bar{\tau} = \mathrm{tr}_g K = \mathrm{const}$, and that $K \neq 0$ in some region of M. Assume further that (M, g) has no conformal Killing vector fields. Then there is a small neighborhood of $\bar{\tau}$ in $W^{1,p}$ such that for any τ in this neighborhood, there exists $(\phi_\tau, W_\tau) \in W^{2,p}_+ \times W^{2,p}$ solving the system (7) for the data $\bar{\sigma}_{ij} = K_{ij} - \frac{\bar{\tau}}{3} g_{ij}$ and τ.

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- Remark: For $K \equiv 0$, one can set $W \equiv 0$, and the system (7) reduces the Yamabe problem.

Conformally Covariant Split System on a Closed Manifold

- Note that the scaling symmetry discussed before can be used to produce new solutions from the already obtained ones. In particular, one can obtain solutions with τ deviating from the vicinity of $\bar{\tau}$, at a cost of rescaling $\bar{\sigma}$. When the seed solution (M, g, K) is a maximal slice, one can also produce new non-CMC initial data with the following scaling argument.

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- **Theorem 2.** Suppose that we already have vacuum initial data (M, g, K) with $\mathrm{tr}_g K = 0$. Suppose $K \neq 0$ for some region. Assume further that (M, g) has no conformal Killing vector fields. Given any $\tau \in W^{1,p}$, there is a positive constant $\eta > 0$ such that for any $\mu \in (0, \eta)$, there exists at least one solution $(\phi, W) \in W^{2,p}_{+} \times W^{2,p}$ of system (7) for the data $(\hat{\sigma} = \mu^{12} K, \hat{\tau} = \mu^{-1}\tau)$.

Conformally Covariant Split System on a Compact Manifold with Boundary

- Let $(M, \widetilde{g})$ be a compact 3-dimensional manifold with the boundary ∂M, and let $\widetilde{\nu}$ be the unit vector normal to ∂M. We assume that $\widetilde{\nu}$ is pointing 'outwards' of M, and therefore to the 'inside' of the black hole. The two null expansions of ∂M are given by

$$\Theta_{\pm} = \mp H_{\widetilde{g}} - K(\widetilde{\nu}, \widetilde{\nu}) + \mathrm{tr}_{\widetilde{g}} K, \;\; H_{\widetilde{g}} = \widetilde{\nabla}_i \widetilde{\nu}^i.$$

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- The condition that ∂M is a marginally trapped surface can be stated as $\Theta_{+} = 0$, $\Theta_{-} \leq 0$. Let us further observe that $\frac{1}{2}(\Theta_{-} + \Theta_{+}) = -K(\widetilde{\nu}, \widetilde{\nu}) + \mathrm{tr}_{\widetilde{g}} K$, and $\frac{1}{2}(\Theta_{-} - \Theta_{+}) = H_{\widetilde{g}}$. Consequently, the condition $\Theta_{+} = 0$ yields

$$\frac{1}{2}\Theta_{-} = -K(\widetilde{\nu}, \widetilde{\nu}) + \mathrm{tr}_{\widetilde{g}} K, \tag{11}$$

$$\frac{1}{2}\Theta_{-} = H_{\widetilde{g}}. \tag{12}$$

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Conformally Covariant Split System on a Compact Manifold with Boundary

- The above conditions can be also expressed in terms of quantities related directly to (M, g).
- Conditions (11) and (12) can be now rewritten as

$$\phi^{-6}\sigma(\nu,\nu) + LW(\nu,\nu) - \frac{2}{3}\tau + \frac{1}{2}\Theta_{-} = 0 \tag{13}$$

and

$$\partial_\nu \phi + \frac{1}{4}H_g\phi - \frac{\Theta_-}{8}\phi^3 = 0.$$

Conformally Covariant Split System on a Compact Manifold with Boundary

- Since Eq. (13) is not sufficient as a boundary condition for W, we will actually replace it with a stronger requirement. Let ξ denote a 1-form tangent to the boundary ∂M. We will require, as a boundary condition, that

$$\phi^{-6}\sigma(\nu,\cdot) + LW(\nu,\cdot) - \frac{2}{3}\tau\nu^\flat + \frac{1}{2}\Theta_-\nu^\flat - \xi = 0, \qquad (14)$$

where $\nu^\flat$ is the 1-form dual to the normal vector field ν. Clearly, Eq. (13) follows from Eq. (14), as $\xi(\nu) = 0$.

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  where $\nu^\flat$ is the 1-form dual to the normal vector field ν. Clearly, Eq. (13) follows from Eq. (14), as $\xi(\nu) = 0$.
- In the remaining part of this section, we always assume that

$$\sigma(\nu,\cdot) = 0 \qquad (15)$$

  on ∂M.

Conformally Covariant Split System on a Compact Manifold with Boundary

In summary, we are now dealing with the following set of equations

$$\Delta\phi - \frac{1}{8}R\phi + \frac{1}{8}|\sigma|^2\phi^{-7} + \frac{1}{4}\langle\sigma, LW\rangle\phi^{-1} - \left(\frac{1}{12}\tau^2 - \frac{1}{8}|LW|^2\right)\phi^5 = 0, \tag{16a}$$

$$\nabla_i(LW)^i_j - \frac{2}{3}\nabla_j\tau + 6(LW)^i_j\nabla_i \log\phi = 0, \tag{16b}$$

$$\partial_\nu\phi + \frac{1}{4}H\phi - \frac{\Theta_-}{8}\phi^3 = 0, \tag{16c}$$

$$LW(\nu, .) - \frac{2}{3}\tau\nu^\flat + \frac{\Theta_-}{2}\nu^\flat - \xi = 0, \tag{16d}$$

where (16c) and (16d) are the boundary conditions on ∂M. Here $g \in W^{2,p}(M)$, $\sigma \in W^{1,p}(M)$, $\tau \in W^{1,p}(M)$, $\Theta_- \in W^{1-\frac{1}{p},p}(\partial M)$, $\Theta_- \leq 0$, and $\xi \in W^{1-\frac{1}{p},p}(\partial M)$ are the assumed data.

Conformally Covariant Split System on a Compact Manifold with Boundary

Theorem 3. Let (M, g, K) be vacuum initial data with boundary ∂M such that $\bar{\tau} = \mathrm{tr}_g K = \frac{3}{2} H = \mathrm{const} \leq 0$, where H denotes the mean curvature of ∂M. Let $\Theta_- = \frac{4}{3}\bar{\tau}$ and $\xi \equiv 0$ so that Eqs. (16) admit a solution $(\bar{\phi} \equiv 1, \bar{W} \equiv 0)$. Assume further that (M, g) has no conformal Killing vector fields, and $K \neq 0$ in some region of M. There is a small neighborhood of $\bar{\tau}$ in $W^{1,p}(M)$ such that for any τ in this neighborhood there exists a solution $(\phi_\tau, W_\tau) \in W^{2,p}_+(M) \times W^{2,p}(M)$ of system (16).

Conformally Covariant Split System on a Compact Manifold with Boundary

Theorem 4. Suppose that (M, g, K) satisfy the vacuum Einstein's constraint equations, and M has a boundary ∂M such that $H \equiv 0$ on ∂M. Assume that $\mathrm{tr}_g K = 0$ and $K \neq 0$ in some region of M. Assume further that (M, g) has no conformal Killing vector fields. Given any data $\tau \in W^{1,p}(M)$, $\Theta_- \in W^{1-\frac{1}{p},p}(\partial M)$, $\Theta_- \leq 0$, and $\xi \in W^{1-\frac{1}{p},p}(\partial M)$, there is a positive constant $\eta > 0$ such that for any $\mu \in (0, \eta)$, there exists at least one solution $(\phi, W) \in W^{2,p}_+(M) \times W^{2,p}(M)$ of the system (16) for the data $(\hat{\sigma} = \mu^{12} K, \hat{\tau} = \mu^{-1}\tau, \Theta_-, \xi)$.

Conformally Covariant Split System with the Cosmological Constant

- The Einstein vacuum constraint equations with a cosmological constant Λ read

$$\widetilde{R} - |K|^2_{\widetilde{g}} + (\mathrm{tr}_{\widetilde{g}} K)^2 - 2\Lambda = 0, \tag{17a}$$

$$\mathrm{div}_{\widetilde{g}} K - d\,\mathrm{tr}_{\widetilde{g}} K = 0. \tag{17b}$$

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- Keeping standard definitions, i.e., LW defined by Eq. (4) and $K = \frac{\tau}{3}\phi^4 g + \phi^{-2}\sigma + \phi^4 LW$, we get the system

$$\Delta\phi - \frac{1}{8}R\phi + \frac{1}{8}|\sigma|^2\phi^{-7} + \frac{1}{4}\langle\sigma, LW\rangle\phi^{-1} - \left(\frac{1}{12}\tau^2 - \frac{1}{8}|LW|^2 - \frac{1}{4}\Lambda\right)\phi^5 = 0, \tag{18a}$$

$$\nabla_i(LW)^i_j - \frac{2}{3}\nabla_j\tau + 6(LW)^i_j\nabla_i \log\phi = 0. \tag{18b}$$

Conformally Covariant Split System with the Cosmological Constant

- Similarly to the scaling symmetry described before, system (18) admits the following scaling. Suppose that system (18) has a solution (ϕ, W). Set $\hat{\phi} = \mu^{-\frac{1}{4}}\phi$, $\hat{W} = \mu^{\frac{1}{2}}W$ for some positive number $\mu \in \mathbb{R}^+$. Then $(\hat{\phi}, \hat{W})$ satisfy system (18) with the data $\hat{\sigma}$, $\hat{\tau}$, and the cosmological constant $\hat{\Lambda}$ given by $\hat{\sigma} = \mu^{-1}\sigma$, $\hat{\tau} = \mu^{\frac{1}{2}}\tau$, $\hat{\Lambda} = \mu\Lambda$.

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- **Theorem 5.** Suppose that we already have vacuum initial data (M, g, K) satisfying Eqs. (17) with $\bar{\tau} = \mathrm{tr}_g K = \mathrm{const.}$ Assume that (M, g) has no conformal Killing vector fields. Assume further that $-|K|^2 + \Lambda \leq 0$ on M, and $-|K|^2 + \Lambda < 0$ in some region of M. There is a small neighborhood of $\bar{\tau}$ in $W^{1,p}$ such that for any τ in this neighborhood, there exists $(\phi_\tau, W_\tau) \in W^{2,p}_+ \times W^{2,p}$ solving system (18) for the data $\bar{\sigma}_{ij} = K_{ij} - \frac{\bar{\tau}}{3}g_{ij}$ and τ.

Conformally Covariant Split System with the Cosmological Constant

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Conformally Covariant Split System with the Cosmological Constant

- Theorem 5 can be generalized in two ways.
- One can generate data corresponding to $\Lambda \neq 0$ from a seed-initial data with $\mathrm{tr}_g K = 0$ and $\Lambda = 0$, i.e., initial data satisfying Eqs. (2).
- The other possibility is to start with seed-initial data that already satisfy constraints (17) with $\mathrm{tr}_g K = 0$ and some nonzero value of Λ. These data can be then used to generate another set of initial data corresponding to some mean curvature $\hat{\tau} \neq 0$ and a different value of the cosmological constant $\hat{\Lambda}$.

Conformally Covariant Split System with the Cosmological Constant

- **Theorem 6.** Suppose that (M, g, K) satisfy the constraint equations (2), and $\mathrm{tr}_g K = 0$. Let $K \neq 0$ in some region, and let (M, g) admit no conformal Killing vector fields. Given any $\tau \in W^{1,p}$ and Λ, there is a positive constant $\eta > 0$ such that for any $\mu \in (0, \eta)$ there exists a solution $(\phi, W) \in W^{2,p}_+ \times W^{2,p}$ of system (18) for the data $\hat{\sigma} = \mu^{12} K$, $\hat{\tau} = \mu^{-1}\tau$, and the cosmological constant $\hat{\Lambda} = \mu^{-2}\Lambda$.

Conformally Covariant Split System with the Cosmological Constant

- **Theorem 7.** Suppose that (M, g, K) satisfy the constraint equations (17) with a non-zero cosmological constant Λ, and $\mathrm{tr}_g K = 0$. Assume that (M, g) admit no conformal Killing vector fields. Assume further that $-|K|^2 + \Lambda \leq 0$ on M and $-|K|^2 + \Lambda < 0$ in some region of M. Given any $\tau \in W^{1,p}$, there is a positive constant $\eta > 0$ such that for any $\mu \in (0, \eta)$ there exists a solution $(\phi, W) \in W^{2,p}_{+} \times W^{2,p}$ of system (18) for the data $\hat{\sigma} = \mu^{12} K$, $\hat{\tau} = \mu^{-1}\tau$, and the cosmological constant $\hat{\Lambda} = \mu^{-12}\Lambda$.

Ideas of Proof of Theorem 1

- First, let us define the operator

$$\mathcal{F}\colon W^{1,p} \times W^{2,p}_{+} \times W^{2,p} \to L^p \times L^p,$$

$$\begin{pmatrix} \tau \\ \phi \\ W \end{pmatrix} \mapsto \begin{pmatrix} \Delta\phi - \frac{1}{8}R\phi + \frac{1}{8}|\bar{\sigma}|^2\phi^{-7} + \frac{1}{4}\langle\bar{\sigma}, LW\rangle\phi^{-1} - \left(\frac{1}{12}\tau^2 - \frac{1}{8}|LW|^2\right)\phi^5 \\ \nabla_i(LW)^i_j - \frac{2}{3}\nabla_j\tau + 6(LW)^i_j\nabla_i \log\phi \end{pmatrix}.$$

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$$\mathcal{F}: W^{1,p} \times W_+^{2,p} \times W^{2,p} \to L^p \times L^p,$$

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- It is easy to see that $\mathcal{F}$ is a C^1-mapping and $\mathcal{F}(\bar{\tau}, \bar{\phi} \equiv 1, \bar{W} \equiv 0) = (0, 0)$. We prove that the partial derivative of $\mathcal{F}$ with respect to the variables (ϕ, W) is an isomorphism at $(\bar{\tau}, \bar{\phi} \equiv 1, \bar{W} \equiv 0)$. The differential at the point $(\bar{\tau}, \bar{\phi} \equiv 1, \bar{W} \equiv 0)$ is given by

$$D\mathcal{F}|_{(\bar{\tau},1,0)}\begin{pmatrix} \delta\phi \\ \delta W \end{pmatrix} = \begin{pmatrix} \Delta - \frac{1}{8}R - \frac{7}{8}|\bar{\sigma}|^2 - \frac{5}{12}\bar{\tau}^2 & , & \frac{1}{4}\langle\bar{\sigma}, L(\cdot)\rangle \\ 0 & , & \Delta_L \end{pmatrix}\begin{pmatrix} \delta\phi \\ \delta W \end{pmatrix}.$$

Ideas of Proof of Theorem 1

- The invertibility of $D\mathcal{F}|_{(\bar{\tau},1,0)}$ follows from the fact that the diagonal terms are invertible. More specifically:
  Claim 1.

$$\begin{aligned} \mathcal{H} \colon W^{2,p} &\to L^p, \\ \delta\phi &\mapsto (\Delta - \frac{1}{8}R - \frac{7}{8}|\bar{\sigma}|^2 - \frac{5}{12}\bar{\tau}^2)\delta\phi \end{aligned}$$

  is invertible and
  Claim 2.

$$\begin{aligned} \Delta_L \colon W^{2,p} &\to L^p, \\ \delta W &\mapsto \Delta_L \delta W \end{aligned}$$

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- The proof of Claim 2 is a consequence of the assumption that (M, g) is closed and has no conformal Killing vector fields.

Ideas of Proof of Theorem 1

- The proof of Claim 1 is as follows. Note that $\mathcal{H}$ is a Fredholm operator of index 0. It suffices to show that $\mathcal{H}$ is injective. Since $(\bar{\phi} \equiv 1, \bar{W} \equiv 0)$ solves the system (7) with the data $\bar{\tau}$ and $\bar{\sigma}$, one has

$$-\frac{1}{8}R + \frac{1}{8}|\bar{\sigma}|^2 - \frac{1}{12}\bar{\tau}^2 = 0.$$

Hence,

$$\Delta - \frac{1}{8}R - \frac{7}{8}|\bar{\sigma}|^2 - \frac{5}{12}\bar{\tau}^2 = \Delta - |\bar{\sigma}|^2 - \frac{1}{3}\bar{\tau}^2 = \Delta - |K|^2.$$

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  Clearly, it is a negatively definite operator.
- Finally, the theorem follows from the implicit function theorem.

Thanks!